Logic I - Session 11

Plan for today

- Damien's comments on quiz
- My comments on teaching feedback
- A bit more on the TF-completeness of SL
- Recap of proof of soundness of SD:

  If $\Gamma \vdash \mathbf{P}$ in SD, then $\Gamma \vDash \mathbf{P}$
- Begin to prove completeness of SD:

  If $\Gamma \vDash \mathbf{P}$, then $\Gamma \vdash \mathbf{P}$ in SD

TF-completeness

- We can express any truth-function in SL.
- Find a sentence that expresses the TF for this TT schema:

T	T	T	A&B
T	F	F	A&~B
F	T	T	~A&B
F	F	F	~A&~B

- We want an iterated disjunction of CSs for the T rows: 1 and 3.
- (A&B) v (~A&B).

TF-completeness

- Strictly, we haven't yet **proven** that SL is TF-complete. We'd need to show that our algorithm always yields a sentence that expresses the truth-function we want. See 6.1E (1d) and 6.2E (1).
- Not only is SL truth-functionally complete, but so is any language that contains formulae TF-equivalent to every sentence of SL.
- E.g. {&,v,~}. (After all, that's all we use in our algorithm!)
- In fact, we can achieve TF-completeness with a single binary connective, '|'.

P	Q	P | Q
T	T	F
T	F	T
F	T	T
F	F	T

TF-completeness with '|'

- To see this, just add a step to our algorithm: translate the old sentence into one that only contains '|'.
- The new one will be equivalent, so it will have the same TT, so it will expresses the same truth-function.
- In our example, our algorithm generated (A&B) v (~A&B).
- To find an equivalent sentence, make replacements in stages.

TF-completeness with '|'

- We start with (A&B)v(~A&B), which is of the form PvQ.
- Now, PvQ iff
  (P|P) | (Q|Q).
- Substitute (A&B) and (~A&B) for P and Q
- (A&B)v(~A&B)
  ((A&B)|(A&B)) | ((~A&B)|(~A&B))
- Now replace the remaining sub-sentences.
- (A&B) iff (A|B)|(A|B). And (~A&B) iff ((A|A)|B)|((A|A)|B).
- So we get:
  ((A|B)|(A|B) | (A|B)|(A|B)) | (((A|A)|B)|((A|A)|B) | ((A|A)|B)|((A|A)|B))

TF-completeness with '|'

- We've just looked at one sentence. We haven't yet **proven** that a language L with just '|' is TF-complete.
- To do that, we need to **prove** that for **any** sentence of SL, there is an equivalent sentence in L.
- Provide an algorithm Z that makes step-by-step replacements like we did. Then prove that:
  - Each step of Z preserves TV, and
  - For any $\mathbf{P}_{SL}$ of SL, Z turns $\mathbf{P}_{SL}$ into a sentence $\mathbf{P}_{L}$ of L.

Soundness of SD

- Basic strategy to show soundness of SD: Use MI to prove that (*) holds for any line n of any SD derivation:
  - (*) If **P**n is the sentence on line n and **P**n is in the scope of only the assumptions in Γn, then Γn $\vDash$ **P**n.
- So for our induction sequence, we use lines of SD derivations.
- For basis clause: (*) holds for n=1.
- For inductive clause: if (*) holds up to line n, it holds for n+1.
  - **P**n+1 had to be justified by applying some SD rule to earlier lines. So, prove for each SD rule X: If **P**n+1 is justified by X and (*) holds up to the nth line, then (*) holds for the n+1st.

Soundness of SD

- (*) If **Pn** is the sentence on line n and **Pn** is in the scope of only the assumptions in Γn, then Γn $\vDash$ **Pn.**

- Most of the proof involves the last step, going through each rule to prove this:
  - For each SD rule X: If **Pn+1** is justified by X and (*) holds up to the nth line, then (*) holds for the n+1st.
- Last time, we went through &E and ~I. Let's do one more: $\supset$I.
- So suppose **Pn+1** is justified by applying $\supset$I, and that (*) holds through line n. Then **Pn+1** is of the form **Qi$\supset$Rk**.
- So, to prove: If **Qi$\supset$Rk** on line n+1 is justified by $\supset$I and is in the scope only of assumptions in Γn+1, then Γn+1 $\vDash$ **Qi$\supset$Rk.**

Soundness of SD

- Since **Qi⊃Rk** is justified by ⊃I, we have a subderivation from an auxiliary assumption **Qi** on line i to **Rk** on line k, where i<k<n+1.
- And since (*) applies for all n < n+1, it applies to i and k.
- So Γk ⊨ **Rk**.
- Now note that since **Qi⊃Rk** on line n+1 is justified by applying ⊃I to the subderivation on i-k, no assumptions in Γk can have been closed before n+1 except **Qi.**
- In other words, every assumption open at k, apart from **Qi**, must still be open at n+1.
- So Γk ⊆ Γn+1 ∪ {**Qi**}.

Soundness of SD

- So far we have:
  - (a) $\Gamma k \subseteq \Gamma n{+}1 \cup \{\mathbf{Qi}\}$, and
  - (b) $\Gamma k \vDash \mathbf{Rk}$.
- Now remember from last time that for any sets $\Gamma 1$ and $\Gamma 2$:
  - If $\Gamma 1 \subseteq \Gamma 2$, then if $\Gamma 1 \vDash \mathbf{S}$, then $\Gamma 2 \vDash \mathbf{S}$.
- So in particular, from (a), we know that since $\Gamma k \subseteq \Gamma n{+}1 \cup \{\mathbf{Qi}\}$:
  - (c) If $\Gamma k \vDash \mathbf{Rk}$ then $\Gamma n{+}1 \cup \{\mathbf{Qi}\} \vDash \mathbf{Rk}$.
- So putting together (b) and (c): $\Gamma n{+}1 \cup \{\mathbf{Qi}\} \vDash \mathbf{Rk}$.
- So $\Gamma n{+}1 \vDash \mathbf{Qi \supset Rk}$. I.e. $\Gamma n{+}1 \vDash \mathbf{Pn{+}1}$.

Completeness of SD

- To prove: If $\Gamma \vDash$ **P**, then $\Gamma \vdash$ **P** (in SD).
- By contraposition, this is equivalent to:
  - $\Gamma \nvdash$ **P** then $\Gamma \nvDash$ **P**.
- So we can assume $\Gamma \nvdash$ **P** and try to prove $\Gamma \nvDash$ **P**.
- We need lots of intermediate steps to do it...
- ...and an important new notion: **maximal consistency**
  - Γ is maximally consistent in SD (MC-SD) iff Γ is consistent in SD and Γ would become inconsistent if **any** additional sentence were added to it.

Plan for proving completeness

- $\Gamma \nvdash$ **P**

(1) ↓

- $\Gamma \cup \{\sim$**P**$\}$ is C-SD

(4) ↓

- $\Gamma \cup \{\sim$**P**$\} \subseteq \Gamma^*$ (for some Γ^* that's MC-SD) (6.4.5)

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(5) →
- For any Γ^* that's MC-SD, Γ^* is TF-C (6.4.8)

(3) ↓

- $\Gamma \cup \{\sim$**P**$\}$ is TF-C

(2) ↓

- $\Gamma \nvDash$ **P**

- $\Gamma \nvdash$ **P** then $\Gamma \nvDash$ **P**.

- If $\Gamma \vDash$ **P**, then $\Gamma \vdash$ **P**.

Completeness of SD

- To prove: If $\Gamma \nvdash$ **P**, then $\Gamma \cup$ {~**P**} is C-SD
    - Suppose $\Gamma \cup$ {~**P**} is NOT C-SD. Then it's inconsistent in SD.
    - Then, by def., some **Q** and ~**Q** are derivable from it.
    - But that means we can derive **Q** and ~**Q** in a sub-derivation from Γ together with the assumption ~**P**.
    - We could then perform ~E on the subderivation, yielding **P**.
    - So we could get **P** in the scope of only the assumptions in Γ.
- So if $\Gamma \cup$ {~**P**} is NOT C-SD, then $\Gamma \vdash$ **P**.
- So if $\Gamma \nvdash$ **P**, then $\Gamma \cup$ {~**P**} is C-SD

Plan for proving completeness

- $\Gamma \nvdash P$

  ↓ ☑

- $\Gamma \cup \{\sim P\}$ is C-SD

  ↓

- $\Gamma \cup \{\sim P\} \subseteq \Gamma^*$ (for some Γ^* that's MC-SD) (6.4.5)

  +

→ For any Γ^* that's MC-SD, Γ^* is TF-C (6.4.8)

  ↓

- $\Gamma \cup \{\sim P\}$ is TF-C

  ↓

- $\Gamma \nvDash P$

- $\Gamma \nvdash P$ then $\Gamma \nvDash P$.

- If $\Gamma \vDash P$, then $\Gamma \vdash P$.

Completeness of SD

- Next, let's prove:
  - If $\Gamma \cup \{\sim \mathbf{P}\}$ is TF-consistent (TF-C), then $\Gamma \nvDash \mathbf{P}$.
- So assume $\Gamma \cup \{\sim \mathbf{P}\}$ is TF-consistent (TF-C).
- By def., there's a TVA that m.e.m. $\Gamma \cup \{\sim \mathbf{P}\}$ true.
- A TVA m.e.m. true $\Gamma \cup \{\sim \mathbf{P}\}$ iff it m.e.m. Γ true and **P** false.
- So there's a TVA that m.e.m. Γ true and **P** false.
- So by def., $\Gamma \vDash \mathbf{P}$ iff there's NO TVA that does that.
- So $\Gamma \nvDash \mathbf{P}$.

Plan for proving completeness

- $\Gamma \nvdash$ **P**

  ↓☑

- $\Gamma \cup$ {~**P**} is C-SD

  ↓

- $\Gamma \cup$ {~**P**} $\subseteq \Gamma^*$ (for some Γ^* that's MC-SD) (6.4.5)

  \+

→ For any Γ^* that's MC-SD, Γ^* is TF-C (6.4.8)

  ↓

- $\Gamma \cup$ {~**P**} is TF-C

  ↓☑

- $\Gamma \nvDash$ **P**

- $\Gamma \nvdash$ **P** then $\Gamma \nvDash$ **P**.

- If $\Gamma \vDash$ **P**, then $\Gamma \vdash$ **P**.

Completeness of SD

- Next, let's prove:
  - If $\Gamma \cup \{\sim P\} \subseteq \Gamma^*$ for some Γ^* that's MC-SD and for any Γ^* that's MC-SD, Γ^* is TF-C, then $\Gamma \cup \{\sim P\}$ is TF-C
- So assume $\Gamma \cup \{\sim P\} \subseteq \Gamma^*$ for some Γ^* that's MC-SD and for any Γ^* that's MC-SD, Γ^* is TF-C.
  - Suppose $\Gamma \cup \{\sim P\}$ is NOT TF-C.
  - Then there's no TVA that m.e.m. $\Gamma \cup \{\sim P\}$ true.
  - But since $\Gamma \cup \{\sim P\} \subseteq \Gamma^*$, any TVA that m.e.m. Γ^* true would also m.e.m. $\Gamma \cup \{\sim P\}$ true.
  - So there's no TVA that m.e.m. Γ^* true. I.e.: Γ^* is NOT TF-C.
  - But since Γ^* is MC-SD, and for any Γ^* that's MC-SD, Γ^* is TF-C, Γ^* is TF-C.
- Our assumption led to a contradiction. So $\Gamma \cup \{\sim P\}$ is TF-C

Plan for proving completeness

- $\Gamma \nvdash P$

  ↓☑

- $\Gamma \cup \{\sim P\}$ is C-SD

  ↓

- $\Gamma \cup \{\sim P\} \subseteq \Gamma^*$ (for some Γ^* that's MC-SD) (6.4.5)

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→ For any Γ^* that's MC-SD, Γ^* is TF-C (6.4.8)

  ↓☑

- $\Gamma \cup \{\sim P\}$ is TF-C

  ↓☑

- $\Gamma \nvDash P$

- $\Gamma \nvdash P$ then $\Gamma \nvDash P$.

- If $\Gamma \vDash P$, then $\Gamma \vdash P$.

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