MASSACHUSETTS INSTITUTE OF TECHNOLOGY

Departments of Electrical Engineering, Mechanical Engineering, and the Harvard-MIT Division of Health Sciences and Technology

6.022J/2.792J/BEH.371J/HST542J: Quantitative Physiology: Organ Transport Systems

PROBLEM SET 5

SOLUTIONS

March 16, 2004

Problem 1

A. The hypodermic needle in the figure below contains a saline solution. If a plunger of area A is pushed in at a steady rate (V), what is the mean exit velocity (V_e) of solution leaving the needle of area A_e? Assume no leakage past the plunger.

Using conservation of mass:

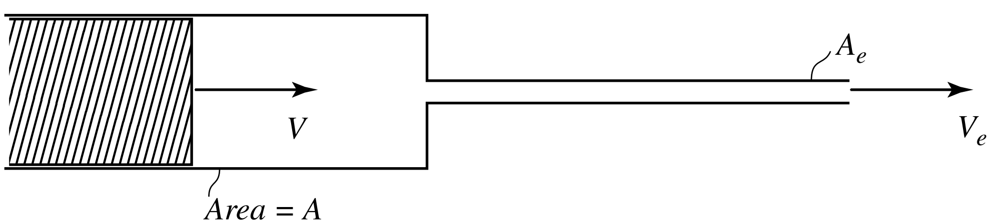
$$\begin{aligned} AV &= A_e V_e \\ V_e &= V \frac{A}{A_e} \end{aligned}$$

B. If there is leakage back past the plunger equal to one-third the volume flow rate from the needle, find an expression for V_e.

If one-third of the needle flow rate leaks back past the plunger we would have:

$$\begin{aligned} AV &= A_e V_e + \frac{1}{3} A_e V_e = \frac{4}{3} A_e V_e \\ V_e &= \frac{3}{4} V \frac{A}{A_e} \end{aligned}$$

C. Neglecting leakage past the plunger, find an expression for the pressure at the face of the plunger if the fluid exits the needle at atmospheric pressure *and* the fluid can be treated as though it were inviscid. The flow can be treated as steady.



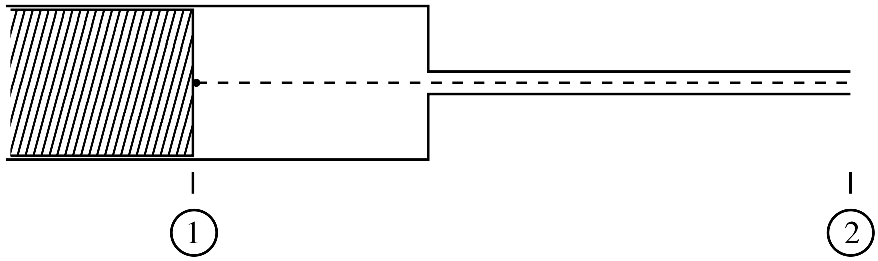
Using Bernoulli's equation between a point on the plunger and the end of the needle:

$$\begin{aligned} P_1 + \frac{1}{2}\rho V_1^2 &= P_{\text{atm}} + \frac{1}{2}\rho V_e^2 \\ P_1 - P_{\text{atm}} &= \frac{1}{2}\rho \left(V_e^2 - V_1^2 \right) \end{aligned}$$

but V_1, V_e were given above in A.

$$P_1 = \frac{1}{2}\rho V^2 \left[\left(\frac{A}{A_e} \right)^2 - 1 \right]$$

assuming $P_{\text{atm}} \equiv 0$



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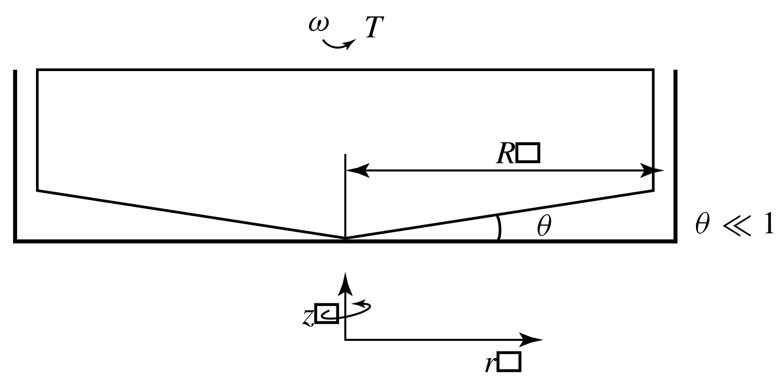
Problem 2

A common type of viscometer consists of a cone rotating against a fixed plate, as shown in Figure 1. Show from physical arguments (or otherwise) that the shear rate is independent of r. (*Hint:* $v_\phi = A(r)z$.) Explain how this viscometer can be used to construct the stress-strain relationship of a non-Newtonian fluid like blood, when the torque T on the cone and the angular speed ω are known.

Figure 1:



In order to construct a flow curve for the unknown fluid we must measure both the shear rate *and the* shear stress *of the fluid. We are given the cone viscometer with angular velocity ω and torque T. The cone angle, θ, is very small, so we make the following approximations:*

$$\tan\theta \approx \sin\theta \approx \theta; \quad \cos\theta \approx 1; \quad R\cos\theta \approx R$$

The angular velocity ω should tell us about shear rate. Let us consider a band of fluid at a distance r from the center that is d*r wide and h high. (See Figure 2.) h is given by the geometry of the device and is*

$$h = r\tan\theta = r\theta$$

The velocity of the cone would be ωr at the selected radius. The shear rate, $\dot{\gamma}$, would then be

$$\dot{\gamma} = \frac{\partial v_\phi}{\partial z} = \frac{\omega r}{r\theta} = \omega\theta$$

Note that $\dot{\gamma}$ is independent of the radius, r.

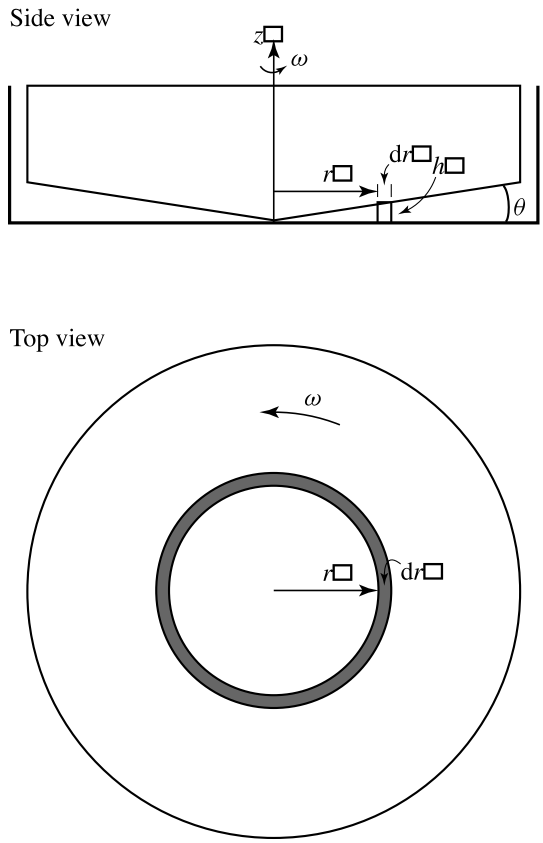
Next we need to relate the applied torque, T, to the shear stress, τ. The shear force *acting on the differential surface ring of width* dr *and radius r would be*

$$\mathrm{d}F = \tau(r)2\pi r\,\mathrm{d}r$$

In our case, $\tau(r)$ is actually not *a function of r.*

$$\tau = \mu\frac{\partial v_\phi}{\partial z} = \frac{\mu\omega}{\theta}$$

Figure 2:



The contribution of $\mathrm{d}F$ *to the torque would then be*

$$\mathrm{d}T = r\,\mathrm{d}F = 2\pi\tau r^2 \mathrm{d}r$$

The total torque would then be

$$T = \int_0^R 2\pi\tau r^2 \mathrm{d}r = 2\pi\tau \int_0^R r^2 \mathrm{d}r = \frac{2\pi\tau R^3}{3}$$

So

$$\tau = \frac{3T}{2\pi R^3}$$

Finally, we can use the measured torque and angular velocity to measure viscosity, μ.

$$\tau = \frac{\mu\omega}{\theta} = \frac{3T}{2\pi R^3}$$

$$\mu = \frac{3T\theta}{2\pi\mu\omega}$$

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Problem 3

Consider laminar viscous flow in a cylindrical vessel. Show that the magnitude of the shear rate at the wall is given by:

$$\dot{\gamma} = \left.\frac{\partial v}{\partial r}\right|_{\text{wall}} = \frac{8\bar{v}}{D}$$

where $\bar{v}$ is the average flow velocity through the vessel and D is the diameter.

$$\dot{\gamma} = \left.\frac{\partial v}{\partial r}\right|_{r=a}$$

The velocity profile for laminar viscous flow (Poiseuille flow) has the parabolic profile

$$u(r) = \left(1 - \frac{r^2}{a^2}\right) u(0)$$

where $u(0)$ is the centerline velocity

$$\begin{aligned}\frac{\mathrm{d}u}{\mathrm{d}r} &= -\frac{2r}{a^2}u(0)\\ \left.\frac{\mathrm{d}v}{\mathrm{d}r}\right|_{r=a} &= -\frac{2}{a}u(0)\end{aligned}$$

Noting that mean velocity, $\bar{v}$, is half the centerline velocity:

$$\begin{aligned}\bar{u} &= \frac{1}{2}u(0)\\ D &= 2a\\ \left.\frac{\mathrm{d}v}{\mathrm{d}r}\right|_{r=a} &= -\frac{8\bar{v}}{D}\end{aligned}$$

The magnitude is simply the absolute value, which is $\boxed{\dot{\gamma} = \left.\frac{\partial v}{\partial r}\right|_{r=a} = \frac{8\bar{v}}{D}}$

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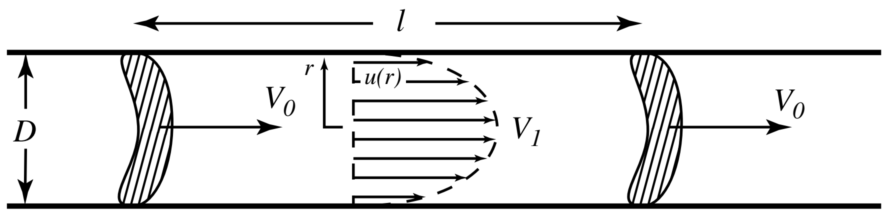
Problem 4

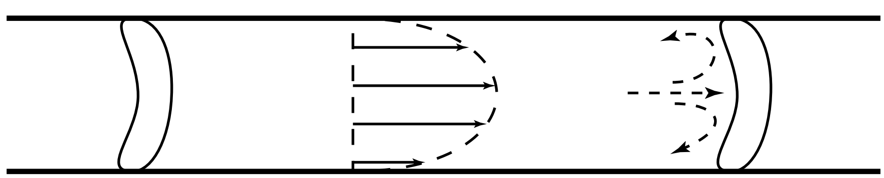
One simple and instructive model of the flow of erythrocytes through the capillaries is shown in the sketch below. The erythrocyte fills the tube so that a bolus of plasma is trapped between each pair of cells and travels with the cells.

If the distance between cells, l, is large compared to the capillary diameter D, the velocity profile in the plasma between the cells is nearly that of a Poiseuille flow. Show that the plasma centerline velocity, V_1, is twice the erythrocyte velocity V_0.



Only when the cells are far apart will the flow be fully developed as given in the problem. Since a bolus of fluid is trapped, the flow rates of fluid near the cells and in the middle must be equal.

$$
\begin{aligned}
u(r) &= v_1\left[1-\left(\frac{2r}{D}\right)^2\right] \qquad \textit{Poiseuille Flow} \\
\iint u(r)\mathrm{d}r\, r\, d\theta &= \pi\left(\frac{D}{2}\right)^2 v_0 \\
2\pi\int_0^{D/2} v_1\left[1-\left(\frac{2r}{D}\right)^2\right] r\,\mathrm{d}r &= \pi\left(\frac{D}{2}\right)^2 v_0 \\
2\pi v_1\left[\frac{1}{2}\left(\frac{D}{2}\right)^2-\left(\frac{2}{D}\right)^2\frac{1}{4}\left(\frac{D}{2}\right)^4\right] &= \pi\left(\frac{D}{2}\right)^2 v_0 \\
v_1 &= 2v_0
\end{aligned}
$$



P.S. The flow behind the red cell is complicated. Nevertheless, the average flow rate must equal v_0.

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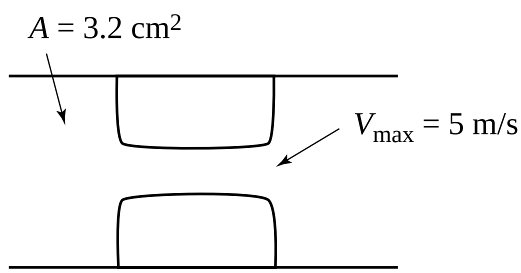
Problem 5

A. A patient has a diseased aortic valve. The valve does not leak, but it has stenosis leading to maximum velocity of 5 meter/sec exiting the valve. If the peak flow rate in systole through the valve is 350 ml/sec and the left ventricular outflow tract area is 3.2 square centimeters, what is the maximal systolic gradient across the valve? What are the assumptions that you made, and why are they reasonable?

Bernoulli:

- *temporal term out* $\rightarrow$ *valid at peak since* $\frac{dv}{dt} = 0$, *so okay*
- *inertia (over shear) dominates* $\rightarrow$ *check Re, know from lecture the Re, aorta huge! So okay*



$$
\begin{aligned}
Q_{\text{peak}} &= 350\text{ml/sec} \\
Q &= A_1 V_1 = A_2 V_2 \\
350\text{ml/sec} &= A_2 (5\text{m/sec})(100\text{cm/m}) \\
A_2 &= .7\text{cm}^2
\end{aligned}
$$

$$
\begin{aligned}
\Delta P &= \frac{1}{2}\rho\left(v_2^2 - v_1^2\right) \\
&= \frac{1}{2}(1)\left((500\text{cm/s})^2 - (109\text{cm/s})^2\right)(\text{dyne/cm}^2) \\
&= 119059.5\text{dyne/cm}^2 = 89.5\text{mmHg}
\end{aligned}
$$

B. Occasional patients have stenosis of the aortic valve, but also have a narrowed left ventricular outflow tract just proximal to the valve. If the maximal systolic velocity exiting the aortic valve stenosis is 5 meters/sec and the flow rate is 350 ml/sec but the outflow tract area is now 1.5 square centimeters, what is the maximal gradient across the valve?

$$\begin{aligned} 350\text{ml/sec} &= A_1 V_1 = (1.5\text{cm}^2) V_1 \\ V_1 &= 233.3\text{cm}^2 \end{aligned}$$

$$\begin{aligned} \Delta P &= \frac{1}{2}\rho\left(v_2^2 - v_1^2\right) \\ &= \frac{1}{2}(1)\left((500\text{cm/s})^2 - (233.3\text{cm/s})^2\right)(\text{dyne/cm}^2) \\ &= 97778\text{dyne/cm}^2 = 73.5\text{mmHg} \end{aligned}$$

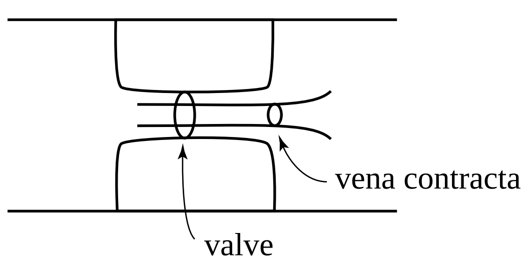
C. For both of the above cases, calculate the area of the vena contracta (the area of the smallest region of the jet). Is the TRUE valve area larger or smaller than the vena contracta area?

case 1

$$\begin{aligned} 350\text{ml/sec} &= A_2(500\text{cm/sec}) \\ A_2 &= .7\text{cm}^2 \end{aligned}$$

case 2

$$\begin{aligned} 350\text{ml/sec} &= A_2(500\text{cm/sec}) \\ A_2 &= .7\text{cm}^2 \end{aligned}$$



The true valve is bigger than the vena contracta → that's why the Gorlin constant is different for different valve geometries.

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