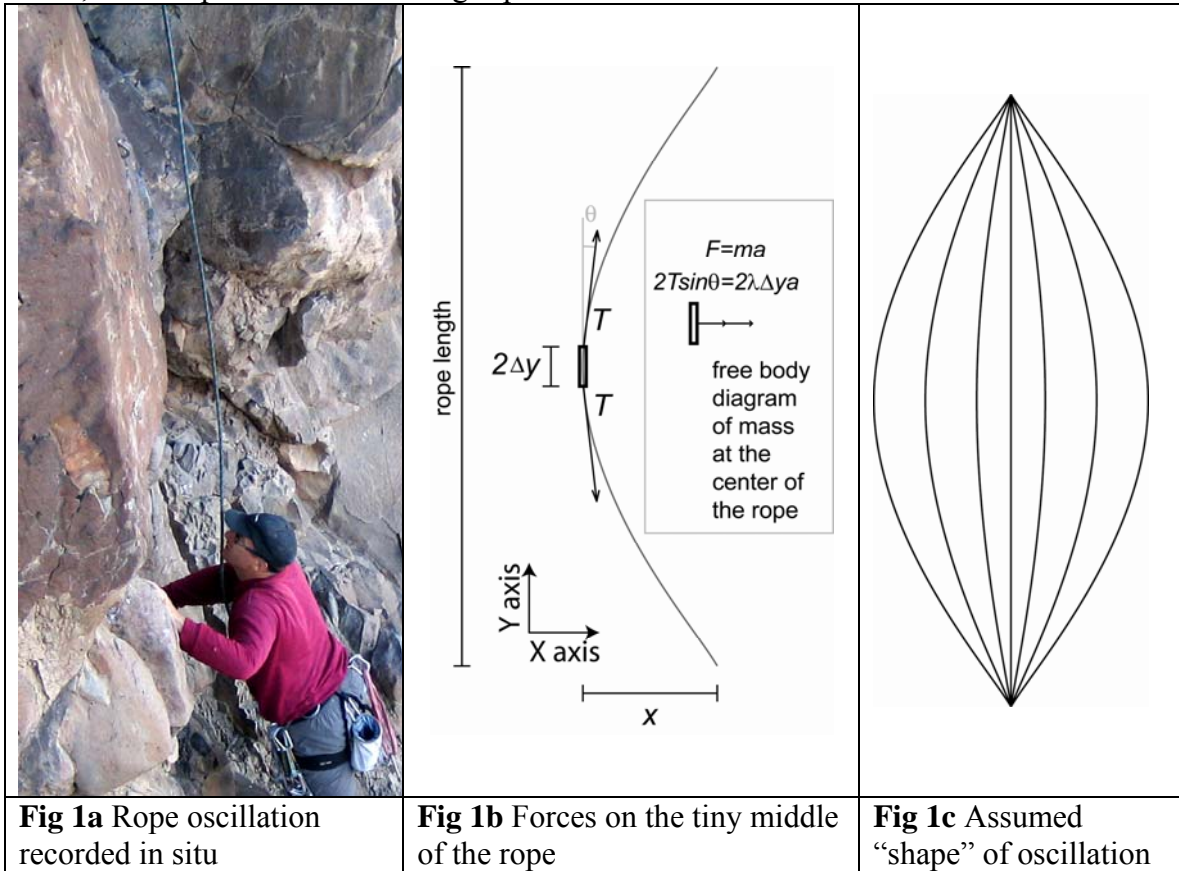


Development of a Simple Model for Transverse Vibration in a Tensioned Rope

Dave Custer's post class thoughts for Feb 13, 2006

Figure 1a shows an oscillation in a top rope. I suspect the photo was taken right as slack was pulled in; as the camera-man, my intention was to photograph the climber, not the oscillation.

The basic premise for calculating the frequency of oscillation is the one that I set up in class is used here. I look at the tiny little piece of the rope in the middle (Figure 1b) and presume that the oscillation is the fundamental or first harmonic (Figure 1c). I invoke Newton's 2nd law, $F=ma$. The distance the middle of the rope is from the vertical line is proportional to the acceleration & the signs are opposite. The math is developed in Table 2. Differences between the result developed in class and the result presented here are due to the assumption in class that the shape of the "plucked" rope is "triangular." Here, the "shape" of the oscillating rope is assumed to be sinusoidal.



Some observations on this text: The genre is not clear, but it might be an example of what you might produce in this seminar, something akin to a problem set solution. And the text might serve as a draft of a proposal; it includes some motivation support and lots of background "model" work. Further, the development says something about how I "think," which is half way between the textbooks that are full on handwaves, without calculus/differential equations, and the textbooks that develop the full wave equation (no handwaving), for which my solution is a special case.

Table 1 Table of Variables for Transverse Rope Oscillations Development

F	The net force on the tiny middle rope section	λ	The linear density of the rope (mass/length)
a or \ddot{x}	The acceleration of the tiny middle rope section	θ	The angle off of vertical at which the tiny middle rope section is pulled by the rest of the rope
T	The tension in the rope	x	The horizontal distance from the rope middle to the vertical line
l	The rope length (presumed not to change much despite the exaggerated difference between the apparent lengths of the arcs in Figure 1c)	Δy	Half the length of the tiny middle rope section

Table 2 Development of Equations for Transverse Oscillation of a Tensioned Rope

$F = ma$	By habit, I know to invoke Newton's 2 nd law.
$2T \sin \theta = 2\lambda\Delta ya$ (1)	As a first pass, in the left/right direction, the two forces on the rope are the left/right components of the tension in the rope; the mass is the length times the linear density. Further trickery (trigonometry and calculus) will be needed to show the relationship between $\sin\theta$ and a .
$x = x_{\max} \sin\left(\pi \frac{y}{l}\right)$	Trickery: The equation that defines the arc of the plucked rope at an instant of time. (I've inverted the usual relationship of the x and y axes for sine waves)
$\sin \theta = \frac{dx}{dy} = \frac{\pi}{l} x \cos\left(\pi \frac{y}{l}\right)$	The slope of the arc is the derivative and, for the points at the end of the tiny rope section, also equal to $\sin\theta$
$\frac{dx}{dy}_{at+\Delta y} = \frac{\pi}{l} x \cos\left(\frac{\pi}{2} + \pi \frac{\Delta y}{l}\right)$	And I want to know the slope at the top and the bottom of the tiny middle rope section, where $y=l/2+\Delta y$ and $y=l/2-\Delta y$. The magnitudes of the slopes/angles are the same for the + and - case, so I include only one.
$\sin \theta = -\frac{\pi}{l} x \sin\left(\pi \frac{\Delta y}{l}\right) \cong -\frac{\pi}{l} x \pi \frac{\Delta y}{l}$ (2)	Trig substitution [$\cos(90^\circ+\theta)=-\sin \theta$] and small angle approximation [$\sin \theta \cong \theta$] produce a useful substitution into Eq. 1.
$-2T \frac{\pi^2}{l^2} \Delta y x = 2\lambda\Delta ya$	And I pulled the minus sign out of a hat. And substituted Eq. 2 in Eq. 1.
$-\frac{T}{\lambda} \frac{\pi^2}{l^2} x = \ddot{x}$	Newton's 2 nd law applied to the middle of the rope, expressed in terms of position (x) and its 2 nd time derivative (\ddot{x})
$\omega = \frac{\pi}{l} \sqrt{\frac{T}{\lambda}}$	The fundamental (angular) frequency ω (in radians/second; to get cycles/second, divide by 2π). For higher harmonics multiply by the integer of your choice (e.g. 2, 3, or 4).

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