ESD.86
Random Incidence
and More

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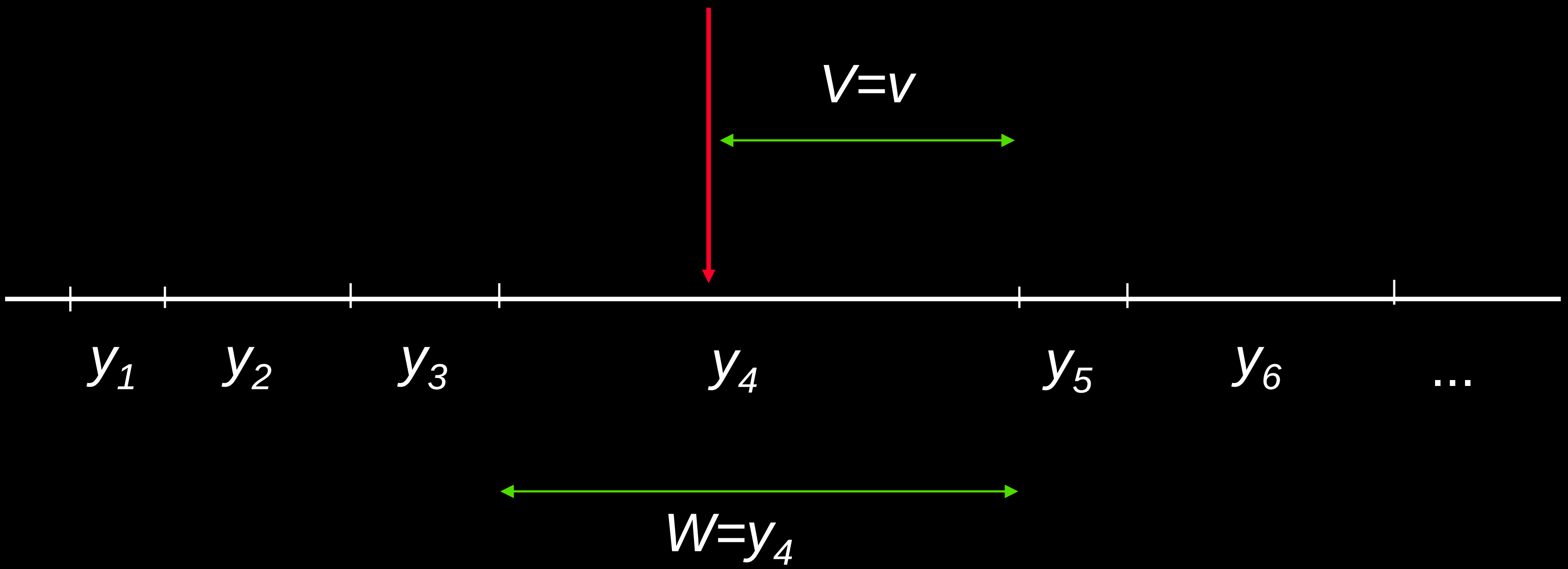
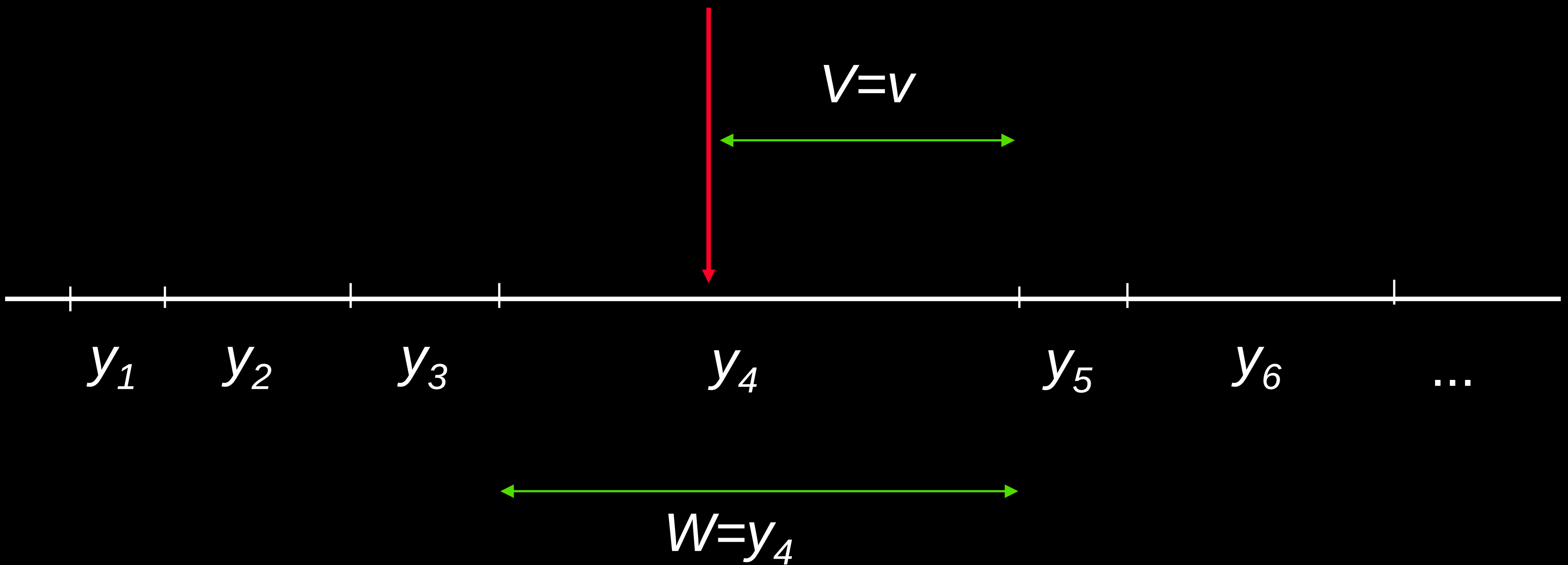


Definitions of the random variables:

Y_i = time interval between the i^{th} and $i + 1^{st}$ arrival event

W = length of the inter-arrival gap in which you fall

V = time remaining in the gap in which you fall



All 3 random variables have probability density functions:

$f_Y(x) = f_{Y_1}(x) = f_{Y_2}(x) = \ldots$

$f_W(w)$

$f_V(y)$

The Inter-Arrival Times

$f_Y(x) = f_{Y_1}(x) = f_{Y_2}(x) = \ldots$

- If the Y_i's are mutually independent then we have a ***renewal process***.
- But the Random Incidence results we are about to obtain do not require that we have a renewal process.

The Gap We Fall Into by Random Incidence

$f_W(w)dw$=P{length of gap is between w and $w+dw$}

$f_W(w)dw$ is proportional to two things:

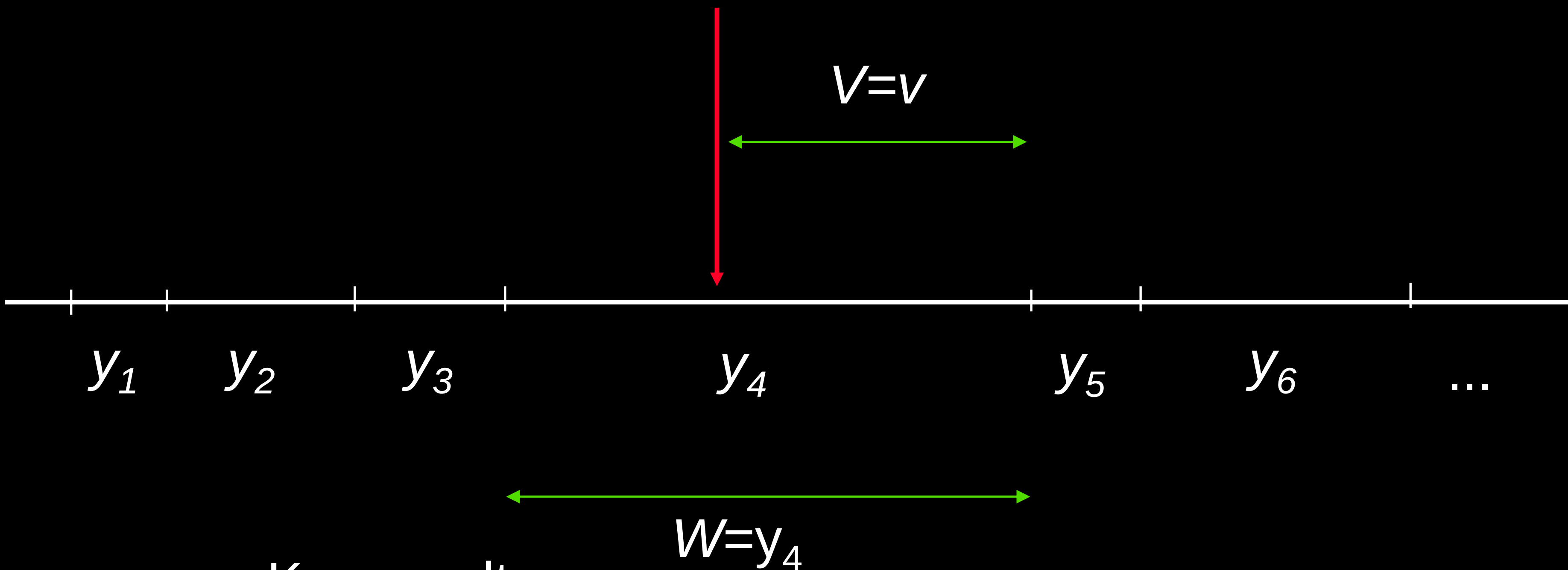
(1) the relative frequency of gaps *[w, w+dw]*

(2) the length of the gap *w (!!).*

Mean Time Until Next Arrival

$$E[V] = \int_0^\infty E[V \mid w] f_W(w) dw$$

$$E[V] = \int_0^\infty (w/2) \frac{w f_Y(w)}{E[Y]} dw$$



Key result:

$$E[V] = (E[Y]/2)(1+ \eta^2)$$

where

$\eta =$ coefficient of variation of the R.V. Y

$E[V] = (E[Y]/2)(1+ \eta^2)$

1. Deterministic Inter-arrivals: $E[Y] = T, \sigma_Y^2 = 0$

$E[V] = E[Y]/2 = T/2.$

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$E[V] = (E[Y]/2)(1+\eta^2)$

4. Suppose $Y = 1.0$ with Probability 0.99

Y=100.0 with Probability 0.01

Then $E[Y]=1(0.99) + 100(0.01)= 1.99=2$

$E[Y^2]=1(0.99) + 10000(0.01)=100.99=101$

$VAR[Y]= E[Y^2]-E^2[Y]=101-4=97$

$\eta^2 =97/4=24.25.$

$E[V] = (E[Y]/2)(1+\eta^2)=(2/2)(1+24.25)$

$E[V] =25.25$ Intuition??

$E[V] = (E[Y]/2)(1+ \eta^2)$

Pedestrian Traffic Light Problem

1st ped. to arrive pushes button.

T_0 minutes later the next Dump occurs.

We are dealing here with a random observer.

$E[Y] = (1/\lambda) + T_0$

$VAR[Y] = (1/\lambda)^2$

$E[V] = (E[Y]/2)(1+ \eta^2)$

$E[V] = (1/2)\,[(1/\lambda) + T_0]\{1+ (1/\lambda)^2 /[(1/\lambda) + T_0]^2\}$

$E[V] = 1/2\lambda + T_0/2 + 1/\{2[\lambda + \lambda^2 T_0]\}$

Time Remaining in the Gap Until Next Arrival

1. Deterministic: $Y = T$ with Probability 1.0

Then $f_V(y)=1/T$, for $0<y<T$.

Suppose T = 10 minutes and event A is:

A = {V>5}.

Then $f_{V|A}(y|A)= f_V(y)/P\{A\}$ for all y in A.

$f_{V|A}(y|A)=(1/10)/(1/2)=1/5$ for $5<y<10$.

Time Remaining in the Gap Until Next Arrival

2. *Y* has negative exponential pdf, mean λ.

We know that $f_v(y) = \lambda e^{(-\lambda y)}$ for $y>0$.

Suppose $\lambda = 1/10$, so that $E[Y]=10$.

Suppose event A:$\{Y>5\}$

Then $f_{v|A}(y|A) = \lambda e^{(-\lambda y)} / P\{A\}$ for all $y>5$.

$P\{A\} = e^{(-5\lambda)}$

$f_{v|A}(y|A) = \lambda e^{(-\lambda[y-5])}$ for $y>5$.

Proves "No Memory" Property

Time Remaining in the Gap Until Next Arrival

$f_V(y)$

Consider $f_{V|W}(y|w)$

We can argue that $f_{V|W}(y|w)=(1/w)$ for $0<y<w$.

So we can write

$$f_V(y)dy = dy\int_y^{\infty} f_{V|W}(y \mid w)\, f_W(w)dw$$

$$f_V(y)dy = dy(1 - P\{Y \le y\})/E[Y]$$

1. For deterministic gaps, $P\{Y \le y\} = \begin{cases} 0 \text{ for } y < T \\ 1 \text{ for } y \ge T \end{cases}$

$$f_V(y)dy = dy(1 - P\{Y \le y\})/E[Y]$$

$$f_V(y)dy = dy/T \text{ for } 0 \le y < T$$

2. For negative exponential gaps

$$f_V(y)dy = dy(1 - P\{Y \le y\})/E[Y]$$

$$f_V(y)dy = dy(1 - [1 - e^{-\lambda y}])/(1/\lambda)$$

$$f_V(y)dy = dy\lambda e^{-\lambda y} \text{ for } y \ge 0$$

Pretend you are a chocolate chip, and you wake up to find yourself in a cookie…..

Photo courtesy of Tom Karlo. http://www.flickr.com/photos/karlo/10746148/

$f_W(w) = w f_Y(w)/E[Y]$

Becomes:

$P_W(w) = w P_Y(w)/E[Y]$,

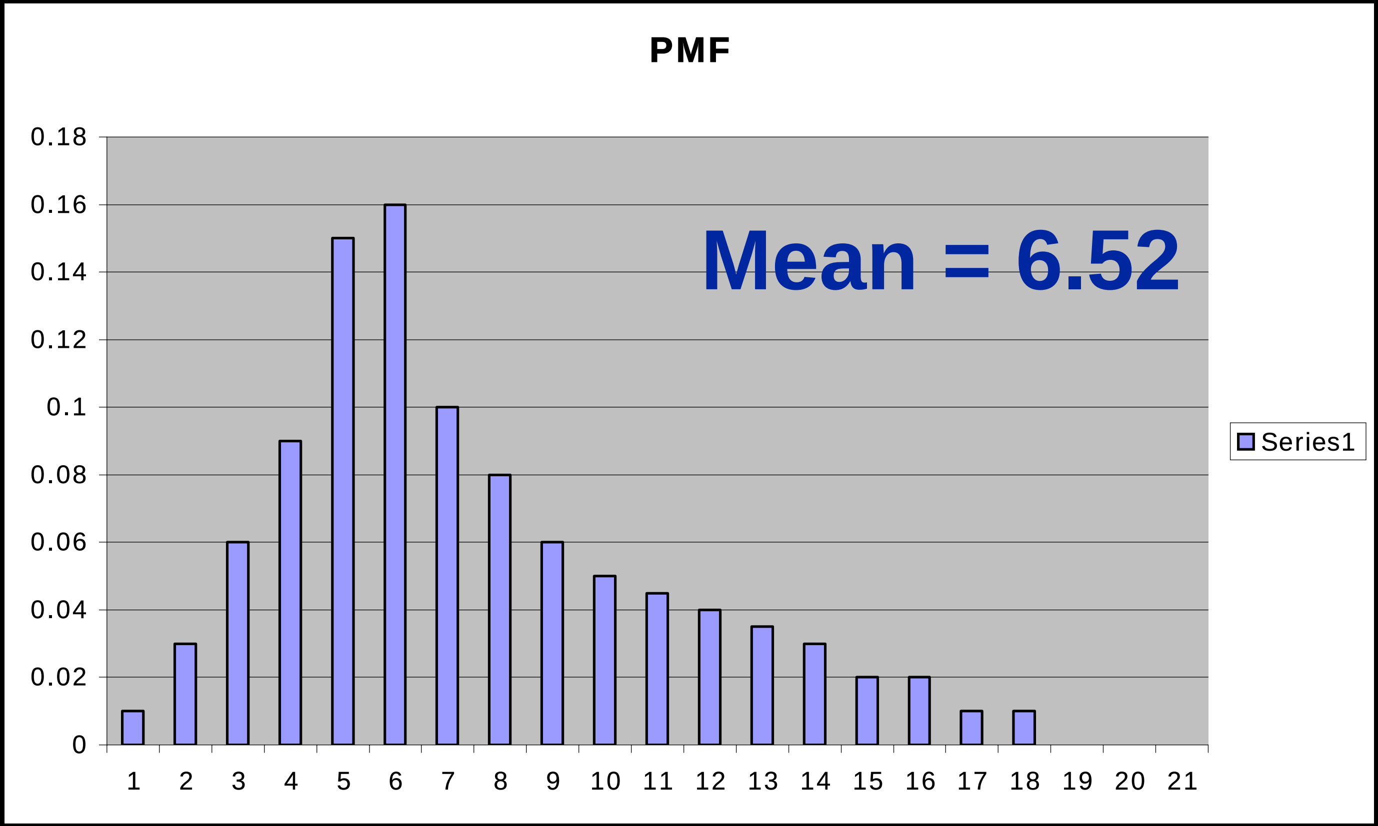
where

Y=Number of chips in a random cookie

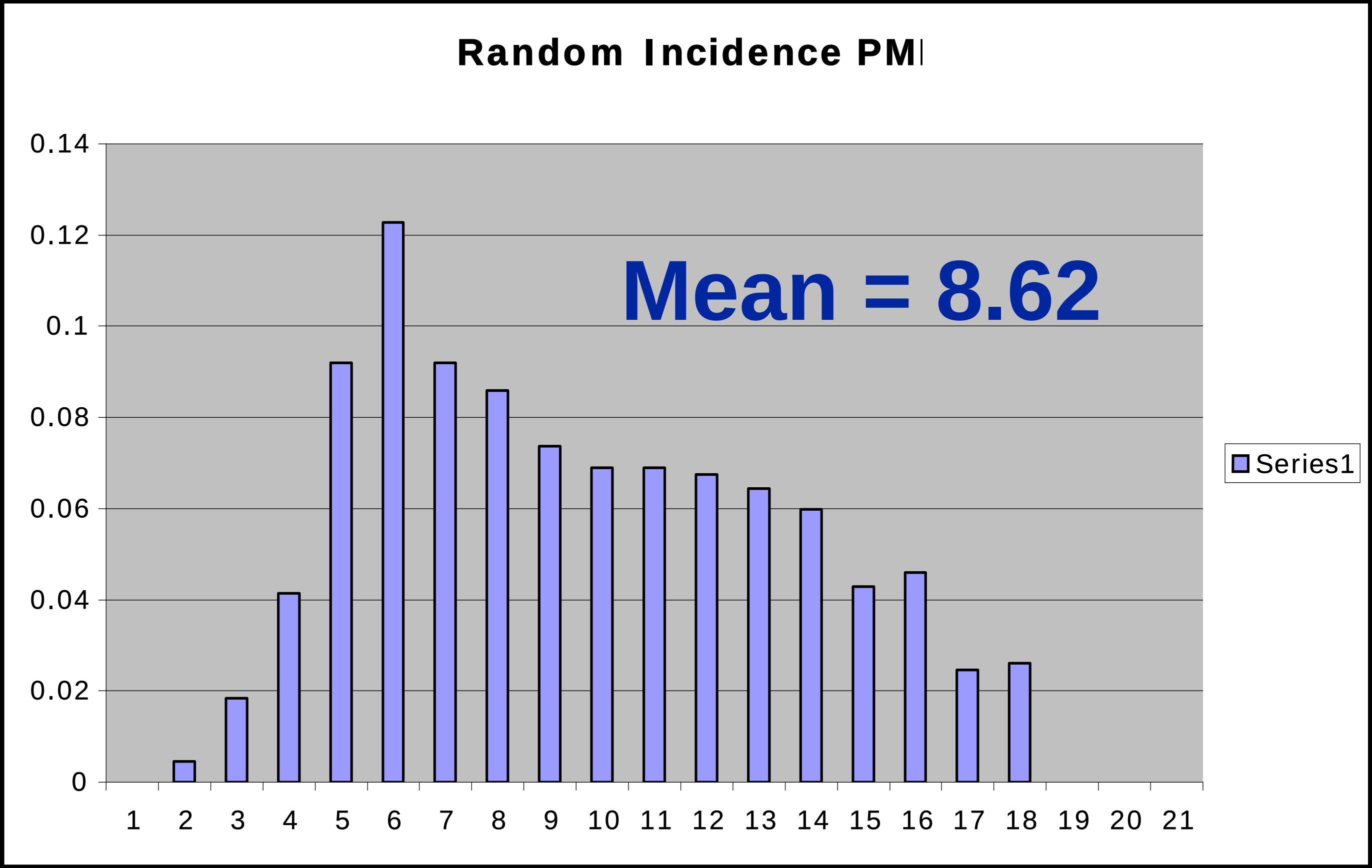
$E[Y]$ = mean number of chips in a random cookie

$P_W(w)$ = P{w chips in a cookie as seen by a random chip within a cookie}

Distribution of Chips in Cookies, By Sampling Random Cookies



Distribution of Chips in Cookies, As Measured by Chips within the Cookies



Where does the chips-in-cookies sampling problem arise in real life?

How About an Infinite Jogging Trail?

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