PROFESSOR: Today we begin the third major part of the class. We've done paper folding, linkage folding, and now we're going to do polyhedron folding. The very last topic we did was hinged dissection, which is somewhere in the middle of all these things. But with polyhedron folding, we're thinking about a two dimensional surface in 3D, something like a cube, and we're interested in cutting along the edges of that shape or somehow cutting along the surface-- that's a good cutting-- and then unfolding into some flat shape. So this is a standard cross unfolding of the cube.

This is the unfolding process, and of course the reverse is the folding process, and both of them are interesting. We're going to start thinking about unfolding. That's one of the most practical problems here. You really want to build some 3D shape out of sheet material. What shape do you cut out in order to bend it into that surface?

So you start with the surface. You need to figure out where to cut so that when you unfold, you get something that has no overlap. If you want to make it from one sheet, it shouldn't have any overlap, and ideally just one piece because then you have to do less welding or edge joining to make that surface. So today and the next couple lectures will be all about unfolding, and then eventually we will turn to the reverse problem, folding problem.

And there are two kinds of unfolding. The one that I just drew is an edge unfolding because it only cuts along edges. Edge unfolding only cut along edges of the surface. This property is nice. If you're building something, you don't want to have visible seems. You have to be bent at the edges, so it's not such a big deal if you're also fusing along an edge.

But in general, you could imagine cutting anywhere on the surface, and this is an example where the solid red lines are cutting on what you see and the dotted red lines were cutting on the backside. And when you unfold that thing, you get this stoplight polygon, and this is what we call a general unfolding, or just unfolding.

There you can cut anywhere. In both cases, we want one piece, no overlap, but you can change where you're allowed to cut. So those are the rules of the game, or two possible rules of the game.

And let me tell you what's known about these kinds of unfoldings. I've mentioned it way back in lecture one. so we can think about edge unfolding, we can think about general unfolding, and we can think about them for convex polyhedra-- simple things like the cube have no dents-- or general polyhedra, non-convex polyhedra. And status is these two corners are open. This one is solved. It can always be done. You can always generally unfold a convex polyhedron, but we don't know about general polyhedra.

Edge unfolding of convex polyhedra, we don't know whether it's possible, but for non-convex polyhedra, we know that's too much to hope for. Not always possible. Both of these questions could go either way, but we know in various generalizations, easy case of convex polyhedra general unfolding, we can do it. The most restrictive, hardest situation is edge unfolding non-convex polyhedra. That's too much. But each of these things, we don't know.

And these are some of the biggest open questions remaining in geometric folding algorithms, this field. I'm going to talk mostly about these three things today, so this open problem and these two results. And then this topic will essentially be next lecture. Obviously it's not solved, but we still have a lot to say. There are various partial results toward solving both of those problems, I would say more on this one. This one is five centuries old. This one is one decade old or so.

So I need to set the stage a little bit before I can talk about any of these things, so let me start with a little bit of terminology. We've in some sense talked about curvature before. I don't think I gave it this name, though, in the context of origami. And I use it almost subconsciously so it's good for me to define it.

If you have some vertex on your polyhedron. Let's say we're looking at a corner of a cube, so this has three 90 degree angles coming together. You sum those angles, we get 270, and then you take 360 minus that sum, and that is your curvature. This is going to be 90.

And what I really care about is whether the curvature is positive, 0, or negative. Positive curvature, which is what we got here-- this is plus 90-- is like a convex cone. So if you're thinking about convex polyhedra, which is one of the situations we care about, you'll always get positive or 0, maybe, curvature vertices, depending on what you're thinking about.

Zero, this is like a piece of paper, so this is what you might call flat, but it's locally flat. It could be drawn flat or we know how to fold pieces of paper. You never change the curvature. The sum of the instant face angles will always stay the same, no matter how you fold this thing. Curvature is an invariant. If you just think locally and forget about the actual folding of the thing, it's kind of flat. So you can think of this as a piece of paper.

We talked about these two situations in the context of Kawasaki's theorem way back when. Kawasaki's theorem applied for both of these for the negative curvature case, which is when you have lots of material all joined at a single point, something like this. Then Kawasaki's theorem changed a little bit. There were some other cases that could happen. So when you have tons of material, this can only happen in a non-convex polyhedron. That's sort of the point. If you're a convex polyhedron, you know you have to have only these two situations.

Where do you get flat vertices? Well, if you think of this point as a vertex and you add up all the angles around it, well, that's 360. 360 minus 360 is 0. So you have zero curvature on all the faces and on the edges. Also, if you look here, you have 180 and other 180. That adds to 360. And then at the actual vertices, that's where you have positive curvature for a convex polyhedron. So just some terminology. Get used to the idea of positive curvature, zero curvature, and negative curvature.

Then we have the idea of a cutting. So cutting is just, what edges do you cut? It's the red stuff in this picture. What edges do you cut in order to unfold? So this is an unfolding, the mapping here, but the red part, I'll call it cutting. Sometimes it's also called an unfolding, but that can be a little confusing.

The main point here is I want to talk about what constraints cutting must satisfy in order to be valid so we can just get a sense of what is happening here. If you look at these pictures, you can see two properties. One is that the cuttings visit all the vertices of the polyhedron. So there's this red stuff. The red stuff is connected. It's acyclic, so it's actually a tree in these pictures, and it's visiting all the corners. Even back here, it's visited on the back.

Is that always true? Most of those properties are mostly true, but it depends a little bit. One thing that is sure is the cutting must span all non-zero curvature vertices. But I do have to say "non-zero curvature." "Spans" just means visits. You have to hit all the vertices of non-zero curvature because anything of non-zero curvature can't be flattened by itself. It needs to be cut. At that point, the cutting has to have degree at least one.

With zero curvature vertices, even if they're not flat in three dimensions, you could flatten them because we know locally, it really is a flat thing. So there may not be any cuts at zero curvature vertices. Most of the time, we won't have to worry about them, but something to think about. Everybody else you have to span.

You can say even more. If you have very negative curvature, if the curvature of a vertex is less than some integer, k times 360 with a minus sign in front, then cutting must have degree strictly more than k plus 1. So if you have negative curvature, you know you have to have at least two cuts for a negative curvature vertex because if you had negative curvature-- one of these things-- you made a single cut. Then all that material is still there. If you then flattened it, it's going to overlap itself because you're trying to flatten it into 360. There's more than 360 material there.

So negative curvature already, you need two vertices. That's the plus 1 here. And as soon as you get to smaller than negative 360, then you have to have at least three cuts and so on, just to partition up into at most 360 groups. This is useful. Negative curvature is basically really tough to unfold and we'll use that to make counter examples here. What else?

your polyhedron has no handles, then cutting has no cycles. A handle is something of higher genus. If this is your polyhedron, we call this a handle. You have sort of a blob down here and you have a connection from one place to the other. If you have a handle, you can have a cycle of cuts like this that does not disconnect the surface, but we basically never think about that situation.

Whenever we're thinking of something of genus zero, like a convex polyhedron or a regular, non-convex polyhedron, you really shouldn't have a cycle in your cutting because if you had a cycle, you'd disconnect your surface into two parts normally, if you're like a sphere or like a disk. That's the acyclic condition and when it holds. Some other good things.

If a polyhedron has no boundary and no handles, and the unfolding has no holes, then cutting is a spanning tree. Spanning tree, it's a concept we've used a few times. It's just a tree. In this case, it's got to visit all the vertices of non-zero curvature and it's a tree, so it's connected and it's acyclic. The main new thing here is that it's connected. We've already said that it should be acyclic with no handles, we've already said that it should span everybody, so it's sort of a summary theorem, but we have a whole bunch of conditions here.

In particular, this will hold if your polyhedron is convex. So for convex polyhedra, you have a spanning tree, and that's what's going on in this picture even for general unfoldings. That's the interesting case. Here, it's maybe more obvious, here it's a little less obvious.

And to sort of see what could go wrong here, I have an example. This is a non-convex polyhedron. There's a whole bunch of views of it up here, and in particular this top view. What we're doing is slicing along just those two edges and some other stuff around the outside, but in particular, we cut those two edges and there aren't any other cuts around there. A polyhedron is set up so that when you open it up, this works out. It's nice and flat. This is extremely rare. If you perturbed this example, it wouldn't hold.

But you can set things up so that, in fact, what's happening is that the total curvature-- this vertex has positive curvature, this has negative, this has positive, and the sum of those three curvatures is zero, and that's what allows this to be flat because the total curvature in that little region is zero, and that's when things are allowed to be flat. So you can cut here and make a separate collection of cuts on the outside, but it's disconnected, so this is kind of weird. Most of the time, things will be connected, and as long as your unfolding has no holes-- and for an unfolding to have holes, you'd have to have this weird property that a bunch of curvatures sum to zero like in that picture-- as long as you don't have that, you're OK.

Let's get to these theorems. Actually, one thing related to this problem, general unfolding of arbitrary polyhedra, is you can't be too general what you mean by non-convex polyhedra. So we have this example. It's the simplest nasty polyhedron there is this. It has one vertex at the top there with a big dot that has negative curvature. There's more than 360 degrees of material from all these triangles. And that's sort of all that it has. This is the polyhedron. This is what we call boundary of the polyhedron. So far, all the polyhedra I've done haven't had that, and you may have seen that I assumed that it wasn't there for that last lemma.

This polyhedron can't be unfolded at all in one piece, no overlap. Why? Because it has one vertex. You could think of these as vertices, but they're kind of flat. Everything's flat except that one point. It has negative curvature. Negative curvature, we said you have to have at least two cuts coming in there. If you just made one cut, then when you flatten this thing, it'll overlap itself locally.

So there's got to be at least two cuts coming in there. I don't think that even matters much. I mean, there's at least one, certainly. Where could those guys go, those cuts? They have to wander around the surface. If they just stop in the middle of nowhere, then you're sort of doing nothing because a cut that just stops in the middle of a flat vertex, well, you might as well not have done that cut, so you could erase it.

So it's got to go somewhere. They could go to each other, in which case you've made a cycle and then you've disconnected your surface because this is like a disk, or they could go to the boundary. And if they both go to the boundary, again, you disconnect your surface, two pieces. So this is kind of pathetic, but you can't unfold this with one piece, no overlap.

So when I say, in this picture, and I say that it's open whether non-convex polyhedra can be generally unfolded, I mean non-convex polyhedra without boundary. I'll even give you handles if you want. I'm not sure that it matters too much, but boundary seems to make a big difference. So what's wrong is this polyhedron is kind of incomplete, and as long as you close it up somehow and don't have these boundary effects, then maybe you can generally unfold everything. That's this question.

That was a little bit on this. We'll come back to it more next lecture. Next, I want to do this one, general unfolding of convex polyhedra. This is really the most positive news I could give you. All this unfolding stuff, it's the one good result. We know several good results, but in terms of that table, it's our only good result. And to do that, there's a bunch of solutions to this problem. They all use the idea of shortest paths.

The shortest path is a path that's shortest. So you have some surface, you have some points. You do this all the time when you're flying between two cities. You follow a shortest path on a sphere. It's not straight in three dimensions. It's the shortest thing subject to lying on that surface, what is the shortest path you can do?

So you're used to it on a sphere. It's a little weirder on a polyhedron, but it's just the same idea. Take all the possible paths you could, find the shortest one. You fix the two endpoints, x and y, and you get to optimize this. Whatever is shortest is the right thing.

I have some pictures of what they look like here. So these are some convex polyhedra. They're probably random points on a sphere, take the convex hull. And then we pick some vertex, or some point x in the corner there, and this is computing the shortest paths from x to every other vertex on the polyhedron.

So they look very straight in this case because the polyhedron is almost a sphere.

They look kind of like great circular arcs on a sphere, but they're not quite. You can look carefully at an edge here, this does bend a little bit. It's a bit subtle. It bends only a little bit because the property you want is if you look at this triangle and this triangle and you unfold them-- so right now, they have some dihedral angle between them, and if you open it out so that they're flat, then this line should be straight.

Still the case that shortest paths are straight lines, but only when you unfold. That's the idea. So if you look at all the triangles are visited by a path, and you unfold them to be a straight thing-- this is called developing, you flatten it out-- then actually, the shortest path will be a straight line, so that makes it really easy to draw these things. There still might be multiple candidates, like to get here, should I go this way or the other way around, but each of those will be locally straight, and this is a property called geodesic.

Shortest paths always unfold straight. And in general, anything that unfolds straight, it might not even be shortest, is a geodesic. These are like locally shortest. Locally, you can't make them any shorter, but they might have made the wrong choice. They might have gone around the wrong way. Geodesic means going straight for a long time. You could actually spiral around and do all sorts of crazy things.

But shortest paths never cross themselves because if you had the shortest path that crossed itself, that wasn't shortest. You should have just gotten rid of this part and gone straight through the crossing. That's sort of trivial.

And there's another good, fun property that you may not have noticed in those figures, but they never pass through positive curvature vertex. So if you look at these pictures, they might end at a positive curvature vertex because that's where we told the shortest path to go, but in the middle, they're always crossing edges. This is a convex polyhedron so everything's positive curvature or zero. In that case, the paths can really only go through zero curvature points. They might start and end wherever, but in between, they never hit a corner.

Why? Because if you have a positive curvature vertex-- I'm just going to sketch this idea-- if you went up here in order to go back down somewhere like there, it's always better to shortcut a little bit and not go through the vertex. It's better to go around one way or the other. I'm just going to wave my hands at that, but it's true. So what?

Shortest paths are going to be a really powerful tool for finding unfoldings. In particular, I want to define the star, that picture, of all shortest paths from one point. And do I want points here? I think I want vertices. That's the picture that we drew.

The interesting thing about all the shortest paths, they all start from the same point, they never hit each other. So not only does a shortest path not hit itself, but if you take many shortest paths, or even two shortest paths from a common starting point, they can't hit each other. It could be one is a subset of another. For example, if I took the shortest path to here and then I also took the shortest path to a point just beyond it, well, one's going to be a prefix of the other. Other than that, they will never cross each other. I'm going to just assert that, not prove it here.

So really, it does look like a star. In fact, you could fill in more shortest paths. You could take the shortest path to here, for example, and it'll fit in nicely, kind of bisect that angle in there. It's a very simple kind of structure. It's just from a point, you have all this stuff going out. They never hit each other except at something called the cut locus, also called the ridge tree. And these are points with non-unique shortest paths to x. So we're fixing some point, x.

So here's a simpler polyhedron. It's just a square-based pyramid. We're picking x to be in the middle of this face, and drawn in these black lines are the shortest paths from x to all the vertices. Here, that's a straight line. This one, if you unfolded it, it would be straight, and there's some similar ones on the back. And I think this is the back in case you want to see. There's one back face that you can't see at all, C, behind B here, and that's what it looks like. There's no shortest paths there.

Now, there's these dashed lines. That's the ridge tree. That's the cut locus. And these are points on the backside with respect to x. If you're going from x, you go around behind to C, or you could go from x around behind to D and then to C. At some point, those things meet and are still of equal length. So here, if you're trying to go to x, you could go around this way or you could go around this way. They will be the same length, and all the points with that property are the dashed lines.

Now, if you're familiar with [? Vornar ?] diagrams, this is the [? Vornar ?] diagram of one point, x. Imagine you plant grass over your polyhedron, you light a fire here, it burns at uniform speed in all directions. Where the fire meets itself on the backside, that is the ridge tree. That is the [? Vornar ?] diagram.

So it's kind of intuitive that it's there. Maybe less obvious is that it's a tree and it's a spanning tree. It hits all the vertices. So it's a natural cutting. That's why it's called the cut locus, and it works really well. And it's called the source unfolding.

Source unfolding goes back to the mid '80s. A bunch of people discovered it for various computational geometry applications. They didn't care about unfolding at the time. And it's kind of obvious after you think about it for a while, trying to solve this general unfolding problem. Once you have this structure, cut along the cut locus, unfold the star of shortest paths from x.

So let's do it for this example. It's the same example on the left. This is the star unfolding. I've just splayed out everything, so at the boundary, I'm cutting at the dashed part. That's the cut locus. Ignore these little dashed lines. Those are just edges, unfolded.

And what's happening is that all the shortest paths are all here. If I look at any shortest path from x on the left diagram, I can map it to a line segment starting from x. x was a flat vertex. It has 360 degrees of material around it. I just chose it to be somewhere in the middle somewhere. Doesn't matter other than that.

And now there's 360 degrees of material on the flat unfolding. That's great. And you pick any direction here, you can map it to a corresponding direction over there on the surface, see where that shortest path would go if you kept going until you were no longer shortest. When you stop being shortest, the edge of that place is the ridge tree, and that's where you cut. You stop your segment there.

So it's actually pretty obvious this thing doesn't overlap because all of these shortest paths are going in different directions from x. What you have is what we call a star shaped polygon around x. Every point on the surface is visible from x because we just sort of unrolled it to be right there. That's the source unfolding. It's a little hard to see the 3D diagrams, but it's actually really easy once you draw those shortest paths.

So the source unfolding is star shaped. We have another unfolding called the star unfolding, which is not star shaped. Very confusing. I mean, what you call one doesn't matter too much. This unfolding was mentioned in 1948 by Alexandrov, who we'll be hearing about more in the future, but wasn't improved to non-overlap until '92. So this one is much less obvious that it doesn't overlap, and I'm not going to prove it here.

But it goes back to this idea of star. Say, OK, this cut locus is nice, but let's focus on the star of shortest paths from one point to all the other vertices. Instead of keeping those paths as the things that you unroll, what if we cut along them? It's another natural thing to try and it turns out to work.

So here, we cut along the star. We've already proved this result, every convex polyhedron is generally unfoldable. Just with that picture, it's very easy. But hey, it's fun to have more. And this is what the star unfolding looks like for the same example.

This is a little less obvious, but if you think about it, the star, the set of all shortest paths to all the vertices, is a spanning tree. It hits all the vertices because we told it to, and it's a tree because it's just all these edges coming together at a point. So it's a natural cutting also. And magically it works. How it works is a little less obvious.

You see here the ridge tree drawn on the unfolding, and you can see all the parts and map the letter E, this is the bottom square, and stays connected. It's just like you reattach everything around the ridge tree and it doesn't overlap. It's quite difficult, I think, to give intuition why this doesn't overlap, but it doesn't. I'll wave my hands at some point when we have the necessary tools to prove it, I can mention how it's done. But we don't have them yet. We'll get them in a couple of lectures, I think.

These results can be extended also in various directions. Let me tell you briefly about some extensions. For a long time, these were the two ways to solve that problem and that was sort of it. It would be nice, for example, to have some general family of unfoldings that includes the star unfolding and the source unfolding and there's something in the middle, but we don't necessarily know what that is. Do you have a question?

AUDIENCE: [INAUDIBLE]?

PROFESSOR: How do you find the creases is your question? You find the creases because you know for every point here where it was on the surface, and if it was on an edge, then it's a crease point. That's the easy way to do it. I don't know if that's a satisfactory answer, but it can be done.

AUDIENCE: Does it matter where you choose x to be?

PROFESSOR: Does it matter where you choose x to be? It will change the unfolding. It will never overlap, but in that sense there's a whole family of star unfoldings, depending on where you move x. There's a whole family of source unfoldings depending on where you move x.

AUDIENCE: Does it make it more efficient, any choice?

PROFESSOR: I mean, some of the unfoldings might have more cuts, some might have less cuts. You're going to get more cuts when you have-- depends how you count. In some sense, there's only n cuts, n shortest paths. But a shortest path might cut over many faces or it might just cut over one face. Depends where x is. If you choose x to be in a nice place, maybe you could get away with fewer cuts and have to do less welding, but there's no theory about that. More questions.

AUDIENCE: So it doesn't matter where you put x. If it works at one spot, it basically works at every spot. [INAUDIBLE]?

PROFESSOR: For convex polyhedra, it works no matter where you put x. For non-convex polyhedra, some x's might work, some might not. Here, let me show you. Here's more star unfoldings. Cool. They're really crazy looking and not really star shaped. There isn't one point that can see everything. They look kind of spiky like a star, but it's quite different. These are random points on a sphere, like take 42 random points on a sphere, take the convex hull, then take a star unfolding.

Here's an example with a non-convex polyhedron. So one thing we hoped for for a little while briefly was that if you only had one vertex of negative curvature, which is this one, maybe if you choose x to be right there and did the star unfolding, because that cuts x into lots of little tiny pieces, then maybe it would unfold about overlap, but it doesn't work. We were destroyed.

In fact, you can show neither of them will work in general for a non-convex polyhedra, so there's no hope of solving this problem with these techniques, at least with these exact algorithms. You might get lucky, but most of the time, I think they won't work.

Currently, x is a point. You can actually let x be-- I guess I'm not going to be safe-- I'm going to call it a geodesic path. There's some restrictions on when this is allowed, but the idea is you have your surface, you take some straight path on the surface, and then you take shortest paths from there. And if you just think about the source unfolding where you keep x intact and unfold from there, the picture is going to look like, well, you have this straight line when you unfold that thing, and then you have this nice star of stuff around it. And under some simple conditions, that will work without overlap.

What's more impressive is that the star unfolding works from a source like that. Again, there are some restrictions on x that I'm not going to define here, but it can be done. These are two very recent papers by O'Rourke and Itoh and Vilku from the last two years. So that's exciting. We now have four general methods for unfolding convex polyhedra. Again, these are big families. You can choose any geodesic path. Maybe some are nicer than others.

You can do higher dimensions. Star unfolding doesn't really make sense in higher dimensions. I don't think I've thought about it much, but source unfolding makes sense. So you have some four polytopes, a little hard to imagine. You take some point and just radiate out from there, unfold like that until you hit yourself, and then you stop, and that works in any dimension. Source unfolding works in any dimension. That's fairly recent, 2003, Miller and Pak.

Another thing you can do is continuous blooming. This is an idea posed by Connolly several years back and then solved last year, I guess. It was presented in Japan. This is about folded states versus folding motions, an issue we have thought about many times in origami. It was easy for any polygonal piece of paper. We could go from unfolded to folded state by a continuous motion without self intersection. For linkages, it was the big deal. It was all about, can we get from a to b?

For convex polyhedra, you can do it, continuous blooming. This is in the middle of an algorithm of continuously blooming the source unfolding. So here's our point x. This is the cube, and the source unfolding of a cube for this point x in the center of a face, these four vertical lines and then a little x on the top side. The algorithm unfolds one edge at a time but in a very specific order. So it ends up unfolding this entire house shape first, and then it's done one edge of this one, and then it's going to fold the other one down and then unfold one, two for the backside, and then one, two for the left side. And you can show that will not self intersect as long as you had a convex polyhedron. It's not obvious but it's true.

There are some other results that if you have any unfolding that doesn't self intersect at the end, you can add some cuts and make it actually continuously bloom. It's kind of like hinged dissections. It may not work by itself, but you'd cut the pieces into smaller pieces and then it will work. That's pretty good news, and source unfolding just works as is. I might talk about that some future lecture.

Not known, for example, whether the star unfolding continuously blooms. That sounds a little scary to me. We don't actually have an unfolding of a convex polyhedron that does not continuously bloom. I think there should be one, but that's an open problem. Is this always possible, or is there some crazy collection of cuts that you cannot escape? That's that, general unfolding of convex polyhedra.

Let's turn to the-- it's not the last topic. We still want to cover both of these. Next topic is edge unfolding of convex polyhedra. Now, this problem, as I said, goes back to 1525, implicitly at least, by this guy, Albrecht Durer, who was a cool guy. This is Renaissance time. He did many different things. I guess painter is maybe his primary profession, but he did a lot of different things. He studied early perspective, all that good stuff.

This is one of his famous prints. It was actually on display at the MFA last year, I got to see it, and it has a little polyhedron thrown in there. It's a nice polyhedron. How did he draw them? Well, he probably built models out of some material. And he was just generally interested in the third dimension and understanding how all these things worked, and so he made a lot of unfoldings in this book.

This is the original title. Here's a translation. Titles were a lot longer in those days. The Painter's Manual is what I usually call it, "A Manual of Measurement of Lines, Areas, and Solids by Means of Compass and Ruler Assembled by Albrecht Durer for the Use of all Lovers of Art with Appropriate Illustrations Arranged to be Printed in the Year MDXXV." And I don't even know what the subtitle is, but there you go. If you read German, I'm told this is a challenge because of the old script.

So unfoldings like this. He did mostly Archimedian solids in this book. This book is several hundred pages long. I have a copy. This is a cuboctahedron unfolded. He is only cutting along edges, and he liked this, I presume, for building models. Here's a fancy one, the snub cube. This one, I think, might have an error. A couple of them have very small errors, but for the most part, he was the inventor of edge unfolding.

Now, he didn't ask, is this always possible, but we did. Mathematicians did in 1976, I believe, or '85. '75, G. C. Shephard, geometer. He was the first one to write it in a paper, does every convex polyhedron have an edge unfolding?

So what can I tell you about this problem? It's hard. Many people have thought about it. I'm not really sure that it's possible. A lot of people do, though. It's possible for a lot of polyhedra. In fact, every polyhedron we've tried to unfold we have eventually unfolded-- convex polyhedron.

Simple set of examples are the Archimedian solids. Doesn't take that much effort to find unfoldings of all of them. You can do this with exhaustive search or just playing around. A lot of people have done this over the years. This is all the unfoldings of those guys.

There's a lot of heuristic software for actually doing this. I think the coolest one right now is Pepakura. If you want to build a paper model of something, you have a 3D model thrown into Pepakura and it'll probably give you a one piece unfolding, even for non-convex shapes. Even though it's not always possible, it'll do edge cuttings, it'll try to do some exhaustive search, and usually does pretty well. Sometimes it'll use multiple pieces. That's the catch. There's a link in the lecture notes.

The most thorough mathematical search is by this guy, Wolfram Schlickenrieder, who wrote the equivalent of a master's thesis in Berlin 13 years ago. I just wrote to him about it yesterday. I was like, hey, this is cool stuff. Can I show it in my lecture? He was like, yeah, it lives on.

He had a class of 10 different algorithms, around that many, possible algorithms for unfolding all convex polyhedra, and then he came up with a dozen different families of polyhedra. I don't have the polyhedra drawn here, just the unfoldings. They're generators of big classes of polyhedra. He applied every algorithm to every class.

Ideally, you get an algorithm that works for all classes or you find a class that foils all algorithms. Sadly, he found neither. Every algorithm was foiled by some example, yet every example was foiled by some algorithm. It's really annoying, but here's some examples where they fail. They just barely fail, just a couple triangles are messed up. Little squares are messed up. I think these are called the turtle polyhedra. It's like a big, flat thing and then a dome on top.

So inconclusive, I guess, is the answer. There was one algorithm that had a degree of freedom. You had to choose a direction, then you cut along all the edges that are most in that direction. And there was a conjecture on the table from this thesis that maybe for every polyhedron, there is a direction that works, but then that was destroyed four years ago. Brendan Lucier from Waterloo proved that that's not true. He found a polyhedron that doesn't work from any direction.

So at the moment, we have no candidate algorithms, which makes me worry whether this could possibly be true. We have some more bad examples. Here are some annoying things, like a cube with a corner cut off. It's a very local thing, but you can mess up really easily. It's actually quite common to mess up, I would say, even though for things like Archimedian, it's much harder to mess up.

Simplest polyhedron that messes up is this sliver tetrahedron. So the polyhedron is drawn at the top. It's a little hard to see, but on the backside, there's an edge like that, if you can imagine. So it's almost flat, so that's why it's drawn flat, and the four corners, like a tetrahedron should have, four faces, the front two triangles and the back two triangles with the line that I drew. And if you unfold it in a simple way, you just cut from a to b, it unfolds like this, no problem. But if you unfold the wrong way, which is to cut both of the diagonal lines, then you end up with the spears that cross each other, so you really have to be careful.

In fact, if you take a random polyhedron-- I choose 80 points on a sphere, I take the convex hull. It's a very nice, round sphere-like polyhedron. And I look at all the unfoldings-- this is probably not all the unfoldings, but I randomly generate unfoldings, and I evaluate, what is the observed probability that I get overlap? And it's very close to 100%.

The conjecture is that as n goes to infinity, the probability of overlap goes to 1. We don't have a proof of that. That would be nice to prove. There's good reason to believe that's easy to prove or that it's true, let's say, that there is a proof. I don't know how to actually formalize it. This is some work from the late '80s.

Just because most unfoldings fail doesn't mean there isn't one unfolding that works. If I were to try to prove that there is an unfolding that works, here's my best hope, and it ties into some things that we've seen with tensegrities. If you take a really big polyhedron, which is the case you worry about, and you look at a small portion of it, that portion will be almost completely flat. Locally, this thing is mostly flat.

Instead of thinking about the big polyhedron, if we at least wanted to get it to work in any patch, think about the case of an almost flat polyhedron. So it's like a little dome. It has boundary now. Makes life a little harder, but let's say it's a nice, convex boundary. This is a convex dome. It's actually polyhedral.

Well, let's make it super, super flat. So we have this z-coordinate. Scale z to squash it down into the plane. What's nice about this is then you can think of your convex polyhedron just as some drawing in the plane. So here maybe is my convex polyhedron.

Now, in fact, each of these vertices is lifted a tiny amount, some infinitesimal amount, but you can think of it in the plane. These are convex faces. We know this thing can be lifted to a convex polyhedron. That lifting is a positive stress on all the edges except the boundary edges. Maybe that stress gives you some useful structure. If there's any hope of this working, that structure better be useful, but I don't know how to use it.

But in particular, open question. If I give you a super shallow, arbitrarily shallow, convex polyhedral dome, can it be unfolded? I don't know. Maybe we'll work on it in the problem session. Haven't thought about it for awhile.

That's if you want to solve everything. What if you just want to solve some special cases? That is a special case motivated by the general case. Well, you can solve the case of a polyhedron with at most six vertices. That's as far as we've gotten.

You can solve some simple examples like pyramids if you take any convex polygon, doesn't have to be irregular or anything, and then you take a point and you take the convex hull. So that's a polyhedron and it unfolds. How does it unfold? Any suggestions where to cut? How?

AUDIENCE: From the top?

PROFESSOR: From the top. Yeah, just cut here. Here, we're only allowed to cut along edges. This is called the volcano unfolding. What do we have? Like that, and then there's just a bunch of triangles. What's nice about the volcano unfolding is if you look at these perpendicular strips here, those triangles will fit inside those perpendicular strips, and therefore they won't intersect each other because these strips don't intersect each other. You could probably even continuously bloom this, no problem. So a very simple example, easy to do with volcanoes. That's the so-called pyramid.

Prism, a little bit more interesting, but not by much. You take some polygon. Again, doesn't have to be regular, but you take two copies of it vertically offset, take the convex hull. This one you can cut in two ways. You could do a volcano-like thing where you cut all the vertical edges and then you cut all but one of the bottom edges. Then you'll have basically a volcano with little rectangles hanging off all the sides. And then there's another copy, the bottom copy of this face, and it just hangs out over here, and you could prove that that won't hit anybody in this very simple situation.

There's another unfolding, though, that I mention because it's useful for other things, and it's so-called band unfolding. Band unfolding, I want to keep intact the radial band around the thing, maybe like this. What do I want to do to keep that edge? Cut, cut. So I cut on all the top edges except one, I cut on all the bottom edges except the same one, and then I also cut this vertical edge. So what I should get is I get my top face, then I get a rectangle, then I get the band that goes around the outside, and then I get another copy of my shape, the bottom side. This is what we call band unfolding.

There's some nice theorems about band unfoldings working out. Let me get to more interesting polyhedra. These are some pretty simple cases, but we don't know very much on the positive side.

Prismoid. Suppose you take a polygon, and you basically inset it. So I want to make a new version of the polygon where all of the angles match, so these edges are parallel to each other. I think, actually, the lengths can change. And then you take the convex hull.

These unfold with the volcano unfolding. It's a little more subtle to prove but it's proved in the textbook. Edges have to stay parallel. I think that might force something. I'm not quite sure, but the constraints are that the edges have to be parallel and the angles have to be equal.

Dome. Dome is actually in some ways simpler. You take some base, and then I want a whole bunch of faces that all touch the base. So in general, it's going to be like a tree of faces, something like this. Every face has to touch an edge of the base. These also unfold, and also with a volcano. This is proved also in the textbook. Both results are by Joe O'Rourke.

Prismatoid. This is the coolest special case I'm going to talk about, the most interesting. I take some convex polygon, and then in a parallel offset, I take some other convex polygon-- no relation to each other except that they're in parallel planes-- and then I take the convex hull, something like that. This one, sadly, is open.

Now, what we do know, and this is related to the band unfolding, if you look at the band around the sides of this prismatoid and just unfold that by itself, that will not self intersect, and that's quite nontrivial, the proof. But if you just unroll the side faces, they will wander around but they won't hit themselves. The only remaining problem is, where does the top and the bottom face go? I think I have an example where that's a little dicey.

Prism. Prismoid. That gives you an answer to your offset. So they can be sheared away from each other, but this is a prismoid. All of the sides are parallel and the angles match, a and b. Here is the volcano unfolding of the prismoid, and here's showing that it's not so obvious that it always works. If you're not careful where you put the top face-- so this is a on top of b in plan-- you could attach a to any of these faces. You have to choose one that's not so inside because it's overlapping here. But if you choose the outermost one, it works and you can prove that.

This is the dome. You can even have overhang and they will unfold, volcano style. You no longer have those nice perpendicular wedges to say everything is disjoined. But it still turns out. Nothing intersects. Here's the prismatoid I wanted to show you. This one's actually almost flat. So a is this triangle right on top of b, the triangle beneath. You take the convex hull and you get all those edges on the outside. And if you're not careful-- this is more volcano style unfolding-- but you can get overlap.

One thing we do know, this is kind of weird. If you try to go more general, and instead of taking a convex polygon on the top and a convex polygon on the bottom, instead you take a smooth convex curve on the top and a smooth convex curve on the bottom, which is getting weird because now there's infinitely many vertices and going to be infinitely many cuts. But there is a natural notion of unfolding which sort of takes the continuum. You take all those lines that go from one side to the other, the rule lines of the convex shape, and you just unfold them.

You don't actually preserve area when you do this. It's not a valid unfolding in the usual sense, but it's a natural generalization of unfolding to smooth shapes. You can prove, well, that part's going to work just fine if you just volcano it, and then you can actually find a place for the top face.

Those are the special cases we know, and even some pretty simple cases that we don't know, although it's almost there. One more open problem. Just mention this because it's a fun problem. So again, I want to edge unfold all convex polyhedra, but I allow multiple pieces to make the problem a little easier.

So let's say I have a polyhedron with f faces. I want to know how few pieces could I get away with? The big open problem is, can I get away with one piece? What if you make it easier? What if I just want, say, little o of f pieces? Smaller than any constant times f. This is open.

What we do know is some constant times f where the constant is less than 1. Best constant I have written here is 1/2. There is a better bound, but I think it's not so easy to summarize in this form. So you can get a little less than half the faces' number of pieces, but that's pretty pathetic. It's a problem that seemed like it would be a good idea, but so far it hasn't seemed to make the problem much easier.

What do I mean by pieces? Well, we're all about one piece unfolding, so now your cutting can have cycles in it and disconnect the surface into multiple parts.

AUDIENCE: [INAUDIBLE].

PROFESSOR: Multiple connected components. The tricky part is to pair up the faces so that everybody has a mate. That's not always possible.

AUDIENCE: [INAUDIBLE]?

PROFESSOR: Yeah. You would hope that one third, but-- I mean, you can do lots of little local arguments and prove this constant, but the big question is, can you get less than a constant?

So last topic for today is edge unfolding non-convex polyhedra. Sort of did that problem addressed, and now this is not always possible. So I want to give you a polyhedron where you cannot edge unfold it. This is actually pretty easy, and we did it way back in '98 when we started working on folding, but it's kind of cheating.

This is a box on a box, and the only edges here are the edges of the two boxes. There's no edges connecting the outside of this face to the inside of that face. That face here is a donut. It's a square donut.

So if you're only cutting along edges, that face is intact. Now I ask you, where does the top box go? The top box has five square faces, not six. There's nothing on the bottom. And somehow, it has to be attached to the rest if I want one piece. That means that all five faces fit in here, but there's only room for one in terms of area so you're screwed.

But this is cheating because what I really wanted to do was generalize this problem, edge unfolding of convex polyhedra. Now, I know they're not going to look convex, but that thing is really not convex. It's really not convex in the sense that you have this face that is not even topologically a disk. It has a hole in it. Convex polyhedra don't have that.

So in some sense, this is a topological problem, you might say. And you might hope that if I looked at polyhedra that are topologically convex, maybe those would unfold by edge cuts. "Topologically convex" means that if you just move the vertices around but preserve the edge structure, then it becomes a convex polyhedron. In other words, I take a convex polyhedron, which we think maybe has an edge unfolding, and then I just pull the vertices around but preserve all the faces. Then can you edge unfold those? And the answer is no.

I have one more example before we get there. This polyhedron, at least all the faces are disks. There's no holes. So it's a cube with little bites taken out of all the edges, same paper. And this thing also does not unfold. It's a little less obvious.

What's cheating about this example is you have two faces, like this purple one and the yellow one, that share two different edges. And for convex polyhedra, two faces either share an edge or a vertex or nothing, but they can't reach around and do to joins. Again, this is not topologically convex, but this is.

So this is two views of the same thing, and if I take these spikes and I just push that vertex down to be really close to this triangle, this will be convex. So it's just a convex polyhedron that I pull on four of the points. Same facial structure. This has no edge unfolding and we're going to prove that.

This example is even stronger in that all the faces are triangles. So what we're going to do is take this thing. It's kind of appropriate for this season. So I've got a tetrahedral spike on top. Think of that as going out of the board. And then in the plane of the board is this triangle, and then I just add in all the edges to make it a triangulation.

Adding edges is a worry because that's where we're allowed to cut, so you really have to worry about all those edges we add. If we add them in that way, I claim, you take this-- we call it the witch's hat-- then you multiply, in some sense, by a tetrahedron, meaning I take four copies of this hat, I put one on each face of the tetrahedron, and you get that example.

Here's one witch's hat and they're just joined along edges to make the tetrahedron. You could do this with bigger polyhedra, too, like octahedron, anything with equilateral triangles, but tetrahedron is the smallest where it works. Just two of them glued face to face would not work, but this way does work. Is that true? Now I've got to think about two of them joined face to face. I don't think it works. Otherwise, we would have done that. Oh yes, I think I see why.

Why doesn't this work? I should say it doesn't work if the spikes are really tall and the base is really flat. I'm going to define what "really" means, but we'll get there. So here's our witch's hat.

When this spike is really tall, these angles, alpha, are very close to 90, a little bit under 90. This angle in the floor here-- well, this angle in the floor is about 60, probably actually is exactly 60. This is an equilateral triangle. So if this thing is very flat, this angle will be almost 300 because it's 360 minus 60. It'll be a little bit less than 300 but almost 300.

These matter. In particular, the total sum of angles here is 300 plus twice 90, which is big. It's bigger than 360. That's the point. In fact, 300 plus one of these angles, 90, would be almost 390, which is way above 360. So this has negative curvature, and even if you cut out one of the spike triangles, it would still have negative curvature.

That's going to be bad news because let's just imagine for the moment. We know unfolding things with boundary is hard, but let's pretend for now that you wanted to unfold a hat in isolation. You wanted to unfold it into one piece without overlap. Think about what you could do. I have to cut with a spanning forest, I guess, meaning it's acyclic, it's got to hit all the vertices. There's only four vertices here. How could I hit all the vertices with a forest?

You may recall from way back when that trees have leaves. Every tree has at least two leaves. Here I might have multiple trees, maybe, but unlikely, and I have to have at least to leaves. Where could those two leaves be? Could they be at any leaves or vertices at degree one? Could they be at any of these three vertices? No, because they have negative curvature. We know a vertex with negative curvature has to have at least two cuts incident to it. You can't stop at these three vertices. That only leaves one vertex and the boundary.

So the only thing you can do if you want to visit all the vertices and start somewhere on the boundary and get to the x, get to the peak. That's all you can do. You couldn't have multiple connections to the boundary because then you would disconnect. Then there's really only two choices and they're reflectionally symmetric.

And we're only allowed to go along edges. It's super constrained. You can go here, walk around, and go up, or you can walk around the other way and go up. Those are actually slightly different because you have two choices of which edge to follow here. They're both screwed because if you look at this point, the white dot, or the white dot up there, what remains here on the outside is almost 300 degrees of material on the base plus one of the 90 degree faces, the back one that you can't see. It's easier to see here. You have the 300 degrees on the bottom and then the white face is still attached to it, and that's almost 390. When you flatten that thing, it's going to overlap itself at that point. Bad news.

So this just says, if I look at a hat in isolation, it can't unfold, but that's not what I care about. I care about four of them joined together. Suppose you had some unfolding of the whole thing, and then I look at, well, what cuts happen within the witch's hat? I know the witch's hat cannot remain in one piece. Therefore, it must be disconnected into multiple pieces, something like this. Again, the cuts have to visit all the vertices somehow, but we know from the perspective of a single hat, that hat must split into two parts.

There are lots of things you could consider. Let's suppose this is possible, not even worry about these kinds of pictures. Well, I claim we have a problem.

Here's a tetrahedron. These vertices are all the same point, actually. I've just unfolded the tetrahedron because it's way easier to draw in two dimensions. So if I look at the hat that is on this triangle, this hat gets disconnected into two parts. There's only three connection points to the rest of the world.

So what these pictures have to look like is they connect two of the vertices by a collection of cuts. If you're going to cut into two halves, you've got to have a path across, and there's only three vertices to visit. There's some collection of cuts that go from, let's say, this vertex to this vertex. At this point, everything's symmetric. Maybe it would be more obvious if I started the center. It could be from here to there, could be from here to there or from here to there, but at least one of those things exist, and by rotational symmetry of this diagram, say it's that one.

Well, what happens to this face? Could there be something like this? No, because then this would be disconnected from the rest of the world. So suddenly, this hat is constrained. Maybe it could look like this. You have a choice. It could look like that or like that, but by reflectional symmetry, same thing. So let's say this one. Remember, x is the same everywhere.

Well, that means this is impossible because this edge is actually glued to this edge, and so then this thing would be disconnected from the rest of the world. So there's a couple of possibilities. It could look like this, or it could look like this. It was one of those two for this face. We've got one face left.

This is impossible because this edge glues to this one, so imagine this thing being picked up and moved over here. So we have a wiggly line here and then this stuff, and so that would be disconnected from the rest of the world, so that can't happen. What about this one? Is that the harder one? I don't know. It's been a while since I've used the argument.

If we have this together with this, that's clearly bad because that forms a cycle. No good. But what if I have this together with this one? It gets hard to see. It's bad. It's a cycle. It starts and ends at x, and if you fold it up right, you'll see there's really two sides to that cycle. It's actually forced, so that's bad.

All right, one more choice, this one. If I do this together with this one, that's going to be bad because that's a cycle. We start and end at x. What if I do this one and that one? Well, that's also bad because here's a cycle that starts at x, ends at x. This and this form the inner of the cycle. All cases are bad, so no edge unfolding. Tragic.

We can think briefly about the case-- do I have time? I have 10 seconds. About the case where there are two triangles. I guess I should draw them like this. So this point is the same as this point. So if I try to simplify this, instead of using four hats, I just use two hats, put one here, one here. Then I could do something like this, I think, and maybe that's OK if there's no cycle formed there. So we really needed the tetrahedron somehow. I think it does work for octahedron and larger also, but just two triangles is not enough.

And that is unfolding. We did not always edge foldable for general polyhedra, even topologically convex polyhedra. Next time, we'll talk more about general unfolding of arbitrary polyhedra.