PROFESSOR: All right. lecture 14 was about two main topics, I guess. We had slender adorned chains, the sort of fatter linkages, and then hinged dissection. Most of our time was actually spent with the slender adornments and proving that that works. But most of our questions today are about hinged dissections because that's kind of the most fun, and there's a lot more to say about them.

So first question is, is there any software for hinged dissections? And the short answer is no, surprisingly. So this would definitely be a cool project possibility. There are a bunch of examples-- let me switch to these on the web-- just sort of random examples, cool dissections people thought were so neat that they wanted to animate them. And so they basically constructed where the coordinates were over time in Mathematica, and then put it on the web as a illustration of that. So this is a equilateral triangle to a hexagon-- a regular hexagon. It's a hinged dissection by Greg Frederickson, and then is drawn by have Rick Mabry.

Here's another one for an equilateral triangle to a pentagon. Pretty cool. And they're hinged in a tree-like fashion even, which is kind of unusual. Greg Frederickson is one of the masters of hinged dissections, and dissections in general. He's probably the master of dissections in general.

And he has three books of different kinds of dissections. This is actually hinged the section on the cover here. The purple and pink pieces hinge like this into a smaller star from the outline of a big star to the interior of a smaller star. And then this star fits nicely inside.

This is another hinged dissection. This book is entirely about hinged dissections, although not just the kinds we've seen. Another kind called twist hinging, which I think this is a twist hinge. The piece flips around the other side. And then there's the third book about a different kind of hinged dissection that's more of a surface hinged dissection where you've got two-- you've got the front and back of this surface and you fold them with like piano hinges with hinges in the plane.

All are very cool books. You should check them out if you want to know more about dissections. They're more about here are cool examples, some design techniques for how to make them. I'll show you one such design technique later on today. but not a ton of theory here, in particular because there wasn't a ton of theory when these books were written. So that's some hinged dissections.

And as I said, cool project would be to make a general tool for animating hinged dissections. There's only a handful out there. Greg has digital files of lots of his hinged dissections. He'd probably be willing to share them, though I haven't talked to him about it. If there was a good engine for animating them I think it would be cool.

Even cooler would be to implement the slender adorn chain business. Take one of these hinged dissections, maybe they just hinge but sometimes there's collisions. But we already know if you refine these guys to be slender, which you can do-- if they are triangulated you can do it with only losing a factor of three in the number of pieces. It would be cool to implement that. And then you can do the slender adorned folding via CDR, which I have an implementation of, or it's not that hard to build one if you have a LP solver. So various project possibilities.

Another cool project would be to just design more hinged dissections. There's still interesting questions. Either use fewer pieces or just make elegant designs. Related to the implementation idea, a particular family of hinged dissections that could be fun to implement are embodied by this alphabet. I showed this in lecture. You can take the letter six and convert it into a square, and convert it into an eight, and convert it into a four, and convert it into a nine via these 128 pieces.

I didn't talk much about this theorem though, so I thought I'd give you a little sketch of how this works. It's actually very simple to construct the folded states of these hinged dissections, and it could be an interesting thing to implement. And it's also just kind of fun.

This is way earlier, 1999, way before we knew that everything was possible. We could at least do all polyominoes of a given size. So let's just think about polyominoes, about polyhexes, polyiamonds, where you have equilateral triangles. And these are called polyabolos for silly reasons, basically, by analogy to a diablo, which is a juggling device.

You can hinge dissect any of them here. You take each square and you cut it into two half-squares, and then you hinge them together like this. This Is 1, 2, 3, 4, 5, 6, 7, 8. So this will make any four-square object, any tetris piece. And generally, you take two end pieces and you can make any anomina.

And the way you prove that that is universal, that it can fold into anything-- it's not so clear from this picture, but it's actually really easy to prove by induction. So the first thing to do in this inductive proof is to check that you can do it for n equals 1. OK. That may sound trivial but this is actually core.

The key property you need in a hinged dissection of a single square into your general family is that there's a hinge visible on every edge of the object. So here, this hinge kind of covers this edge, it covers that edge. So both of these edges have hinges on them, and the other two edges have a hinge on them. They happen to be shared hinges but that's OK. And each of these, that's true.

The triangle, it's a little more awkward. You actually need two hinges to cover the three sides. But you only need these two pieces. One of them is non-convex. It may be hard to fold continuously but you'd refine it if you wanted to do slender adornments. So let's not worry about continuous motion yet. So that's the base case of the induction. How do I do it for n equals 1?

Now inductively, if I have some shape I want to build I'll take what I call the dual graph of that shape. So make a vertex for every square, connect them together if they share an edge-- the squares share an edge-- and then look at a spanning tree of that shape. So just cut some of these edges until you have tree connectivity among those squares.

Every tree has at least two leaves, except in the fall, but every mathematical tree has at least two leaves. Like this is a leaf, if I cut here this would also be a leaf. Leaf is a degree one vertex. So that's a square that only shares one side. So pluck off that leaf, remove that square. The resulting n minus 1 square is, by assumption, can be made by this hinged dissection with two times n minus 1 pieces.

So now we just have to attach this guy on. And here's a figure for that down at the bottom. This is the same thing for triangles, and polyabolos are in the upper right.

So you have some existing hinged dissection, you don't really know what it's like, and you want to add this leaf back on so it shares one edge with one guy. Now this guy could be oriented this way, or it could be oriented this way, but it's the same by reflection. So let's say it's oriented this way.

We know the square is made up by two half-squares, by induction, and so we know that there's a hinge here. Now this hinge connects to some things, in this case, to some t prime. It could be here, it could be up here. And all we do is stick s on here. Now s can rotate.

We have our solution for one guy, and there's two different orientations for him. We're going to choose this orientation because it puts this hinge right there. And so once we do that, normally this would be a cycle, and this thing would be a cycle through here, but we just redo the hinges in here so that the cycle gets bigger.

And the important thing to verify is that the orientations to the triangles are the same, just like the hinged dissection picture I showed. We always go from the base edge to the next base edge, to the next base edge of these right isosceles triangles, and all the triangles are on the outside of the cycle. So we actually construct a cyclic hinged dissection.

Then at the end you could break it and make it a path. And this one is even slender. Remember, right triangles are slender, barely. You can look at all the inward normals. They hit the base edge. So this will even move continuously if it's an open chain. For closed chains we don't know. So that's polyominoes.

Polyaimonds are similar. Pretty much the same thing. You just-- in this case, you might have hinges on both sides, but you rotate this thing so one of the hinges lines up, and you just reconnect the hinges. And it's not hard to show you can always do that, the hinges will never cross. And this proves that these folded states exist, and then we use the slender stuff to do continuous motions. Actually, when this paper was written we didn't have slender adornments back in '99, even in 2005 when the journal version appeared. So it's only now that we know that motions are possible by, in this case directly, in this case with some refinement.

So I thought that would just be fun to see. You can do some other crazy things. So this is a hinged dissection from any four-iamond. So this is four equilateral triangles joined together to any four amino. This is a tetris piece. It's essentially a superposition of this idea with-- you see in here, these four lines make the hinged dissection of Dudeney, from 1902, from an equilateral triangle to a square.

And with some extra stuff added in-- this is maybe a foreshadowing of the idea of refinement, although we didn't really realize it at the time. We want to add some hinges so that we have hinges on the midpoints of the edges instead of the corners. That turns out to be a bit more efficient in this case. So we add some hinging, still hingeable individually, but now we have hinges at the corners and so-- at the midpoints, and we'll have the same property over here. And it allows you to hinge these together. Actually, here it looks like some of them are at the corners not the midpoints. So it's a bit messy.

In general, we can prove if you have any shape and you want to make poly that shape-- so let's call this shape x, you want to make polyexes-- you can do it as long as the copies of the x are only rotated and they're joined at corresponding edges. So if you check, this guy's just been rotated 180 degrees.

In general you can join these things together at matching edges. And the basic technique is just subdivide the thing, triangulate, draw in the dual of the triangulation, and then connect to the midpoints of the edges. And you can show, basically, instead of the hinged dissection going around like this you can just make it go around like this and come back this way. And if you check the sequence of pieces they could visit, it's identical if you go around this way or if you go around this way. And that's enough to show that any folded state is valid

With the triangles and the squares we're essentially exploiting the symmetry of these pieces. So you can rotate them to make them compatible. Here they're forced to be compatible by assuming we only join matching edges. So that was the 2D polyform paper. You can see Frederickson was one of the authors.

In 3D, here's easy way to generalize that. If you take, for example, a tetrahedron, a regular tetrahedron, you take the centroid and cut everything to the centroid. And you end up cutting your tetrahedron, it has four sides, into four of these more slender tetrahedra. And then you take four of them and join them together in this way. You do have to be careful in the way that you join them because, again, on every face we want to have an incident hinge. So we've got to take care in the way that you hinge together to make sure that that is the case.

But it's also cyclically hinged. This gets joined to that. And basically, the same inductive proof works. You just pluck off a leaf, show that you can turn the thing so that one of the hinges aligns with the inductive construction, and then just join the hinges across instead of within the cycles. So pretty easy.

What are we talking about? Hinged dissections software, I guess. Those would be fun things to implement. They've never been implemented, and especially, to see them folding. I thought I'd show you a little bit about hinged dissection hardware, different ways you could make them physically real.

This is kind of mesoscale I'd call it. This is at one centimeter bar, so not super tiny, but I think this could scale down quite a bit. We have a Petri dish here with some liquid in it, if you could read up there. Maybe this is the coolest example. We have a square made up of four pieces, and you add a little bit of salt to that liquid and it pops into the equilateral triangle configuration. So it's sort of spontaneously folding, hinging. Essentially these pieces are slanted a little bit and they prefer-- one weighting causes them to fold one way but when you add the salt they end up flopping the other way.

You could see they're a little bit inexact because of that, but pretty awesome the kinds of hinged dissections. You can get them all to actuate even without much room to do so. This is done at Harvard, George Whiteside's group, chemistry. Kind of related, it's not exactly hinged dissections, but I feel like it's the same spirit, is this idea of DNA origami, it's called, where you take one big strand of DNA and you force it to fold into a particular shape. Here we're folding it into a happy face.

The way that's done is you add in a bunch of little pieces of DNA. So this string, basically, has a-- this DNA strand has a random string written on it basically, and you identify, oh, I want these guys to glue together. So you take this piece of the random string, and this piece of the random string, and you construct a piece of DNA that has both of those, like a little zipper to cause those to zip up. You do that all over the place.

And there's now automatic tools to do this, it's really easy to make DNA origami. It, basically, always works. There's a limit to how big this thing can be because the main strand here is a single piece of DNA, and those are hard to make super big, at least currently. But you get some really nice happy faces and mass produce them. Hundred-nanometer scale. It's kind of like hinged dissection because that strand of DNA is moving, it's actually more like a fixed angle chain, kind of like a hinged dissection. And we're essentially using here universality of hinged dissections of something like polyominoes, though the shapes are a little bit more awkward.

And they've made a maps of the world. You could do two-color patterns, make snowflakes, the word DNA, and crazy stuff. So it was started by Paul Rothman, though a lot of people do DNA origami these days. Cool.

Next paper I wanted to show you-- this is fairly recent-- and it's about getting continuous motions, in particular, in 3D, of hinged dissection-like things. So here we have a chain of balls. These are more like ball and socket joints. So you can maybe see them better here. There's a member going in from the green guy into the center of the red guy, and there's a slot, and the red guy can fold around the-- or the blue guy can fold around the red guy.

And the question is, OK, this is great. You can prove universality, you can make any shape. You just subdivide your dog, or whatever, into two by two by two squarelets and then we know how to connect those together to make a nice Hamiltonian cycle that visits everything. But can you actually fold a chain of balls like this into that dog? And the answer is always yes.

Essentially, you feed a big string of these balls into-- that's actually what's happening in this animation here, although it's a little hard to tell-- you're feeding in, say, at one of the legs, one of the extreme points in some direction, this chain of balls. And as they go in they just start tracking along the path. And you just need to check that you can track along the path. As this guy goes into a corner, for example, you can actually navigate the corner while, at all times, staying within the tube. If you can stay within the tube you know you won't collide with the rest of the chain because this tube is non self-intersecting.

And so the 2D version is fairly easy. This is just circles. A little trickier to check that it actually is possible, with just one turn, with a U-turn, and with a kind of-- I don't know what you call this, not a U-turn-- where you change in two directions-- two dimensions all at once. All of these are possible with this particular mechanism, whatever mechanism you have. If it can do this then you can make anything.

So that's another way to prove motions exist for this kind of polyform special case. Why do we care about this? For building robots.

So these are somewhat different mechanisms, but I have two examples built here at the MIT Center for Bits and Atoms over in the Media Lab with Neil Gershenfeld and many, many people. So you get some idea-- this is a fairly small guy. I mean, the actual size is about this big. You see some feet in the background to give you some sense of scale. It's not very many pieces, but if you made a really long chain it would really be able to fold into anything you want, just servos to make the turns here.

This is a much larger one. The right version is folding. And you get some idea of scale here, this is, when it's fully extended, 144-- should that be feet or inches? It's really big. So a little bit slower, of course, because it has to move a lot more, and it's also quite a bit longer. This is built, in particular, by Skylar Tibbits here.

So that's the idea of robots. In general, we like to make robots that can change their shape. We've seen sheet folding robots, but these are more chain folding robots inspired by proteins, and DNA, and things like that, sort of big versions of DNA origami. What's cool about them is that they stay connected throughout the motion. You can keep your wiring, and you can keep your batteries, and whatnot, and your communication channels connected in this kind of scenario.

This is, by contrast, to more common approaches to reconfigurable robots. You have individual units and they can attach and detached from each other. You could see like these guys picking up blocks, moving stuff around. It's definitely cool, but in practice it's a lot harder to build these kinds of robots because the attach detach mechanism, it's hard to get them to align perfectly, it's hard to get the electrical connectivity. Every piece has to have a battery instead of like every 10th piece, or one battery to drive everything, or tethering, or whatever.

You can do some very cool things and there's a lot of algorithms around for doing this. Daniella Rus, here at MIT, built this robot, and a bunch of others. There's also a very cool theory about these. I've worked on them. You can prove, for example, that all of these models can simulate each other up to constant factors in scale.

So you can take your favorite robot in a molecube and simulate a crystalline robot, or vice versa. And then there's efficient algorithms to-- these crystalline robots, they can just expand and contract and detection and attach. And you can prove that given two configurations you can change it from one to the other up to some scale factor. You can even do it extremely fast in log n time if all the robots are actuating all at once. Anyway, there's cool stuff about reconfigurable robots, but the hinged dissections offers an alternative where everything stays connected at all times, but closely related. I think that was the hardware story.

So we go back to our proof of hinged dissections and why it works. And one of the-- I was kind of surprised I didn't show this in lecture, but I don't remember why I didn't. One missing piece with-- how do you go from a rectangle of one size to a rectangle of another?

You may recall, we had a triangle, we triangulated our polygons so we ended up with some arbitrary triangles. Then we cut parallel to the base halfway up. You can put this over here, put this over here, you get a rectangle of some unknown height. And then to make it universal we wanted to convert everything into a rectangle of height epsilon so that then we could just string them together-- obviously, the area has to be preserved here. If we string together all the epsilon height rectangles we've got one super long epsilon height rectangle. And then we overlay the two dissections. This is how we did dissections.

But how do you do this step from one rectangle to another? This is a very old dissection, at least 1778. It wasn't published by Montucla but he's credited in this publication, and this is Frederickson's diagram of it.

So you take the fatter rectangle and then you take the longer rectangle and you-- first, you make multiple copies of the fat rectangle, just sort of tile strip of the plane to the right. And then you angle the thin rectangle, slightly. First of all, you line up these corners so the top left corners line up, and then we want the top right corner of the thin rectangle to lie on this bottom line.

Turns out this always works. It's not totally obvious but, essentially, these copies of the rectangle you can kind of fold them up. And when you go off the right edge here, you're essentially coming back on the left edge here. And then you're going this way, and you're going this way, and this little piece is exactly the same as this little piece.

And from that you get a dissection. It's not hinged, but you can see that this big rectangle has the tiny piece here, which conveniently fits right over there. It's like a wrap around in the other direction. And then this piece-- well, everything matches up here.

The only other weird thing is this bottom-- when you go below the bottom you also wrap around to the top. And just check all the pieces match up, and you've got your dissection. It's kind of crazy. You have to check this works for all parameters, but it does. And in general, of course, if you have a very long rectangle you need many pieces, relative to the fat one, but that's essentially optimal. OK.

For fun-- this is a general technique called the piece lie technique, or superposing two tessellation of your shape. You can use that same technique, for example, to get the hinged dissection from a regular square to the equilateral triangle. You just angle it right so that, for example, this midpoint hits this midpoint, and various other alignments happen, like this midpoint falls on that edge.

And if you look at it right these cuts give you the four pieces for the square to-- I guess you can see it right here, here are the four pieces of the square. And if you check, everything matches up. You can also make equilateral triangle. In this case, it happens to be hinged. That doesn't always happen. It's a little tricky to tell, maybe, but with practice you can see it.

I mentioned, at some point, that you could take this and turn it into a table that either has four sides or has three sides. One of the annoying things about the table is that you need legs on each of the pieces. So Frederickson was playing around with this fairly recently, in 2008, and he came up with this alternative way of-- essentially the same technique, but you end up with one big piece and lots of smaller pieces. So the idea is you just have a big leg, or a bunch of legs, under one piece of the table. And so this is what the dissection looks like.

Unfortunately, it's not hingeable. But if you add in a couple pieces you can make it hingeable. So at this point, the universality result was probably none. This is actually a lot easier than the way we do it, specialized to this kind of scenario.

This hinges, I think, something like this-- maybe even an animation of it? Yeah. Drawn by Frederickson. So you could see a careful orchestration here just to make sure that, indeed, you can avoid collision. And so that's the proposed table. No one has built it. Another project would be to build some hinged dissections, for example this one, as real furniture. It would be pretty neat.

I have a couple examples here of real furniture built. This is the Dudeney dissection, a four-piece kind of a cabinet. It's got lots of shelves. It looks really practical.

And I don't know the bottom. It looks like there's a bunch of wheels down there. Definitely, you have to have a bunch of table legs in this case. But you can really reconfiguration it in all sorts of ways. The close up. That looks pretty cool.

It's made by D Haus Company. Any German speakers? Anyone know what "haus" means? Same in English, house. So they actually built a house. And I can't tell whether this is a real building or a very good computer rendering. It may be real.

AUDIENCE: [INAUDIBLE]

PROFESSOR: What's that?

AUDIENCE: It looks like a rendering.

PROFESSOR: It looks like a rendering. Yeah. At some point later they have people walking by, but it could be a composition. Anyway, it's an idea of having a house for any season. You can reconfigure it dynamically with these tracks. It's a pretty cool idea. It would be neat to experiment with. Anyway, hinged dissections in practice.

It's funny to take a 2D dissection, but, I think, in architectural setting you can't change where the floor is. So probably, 2D dissection makes sense. There's the real, maybe real version? I don't know. So that was rectangular rectangle. OK. I'm cheating a little bit.

Another question. This is a very specific question, but for step three, which is where we did all the action of rehinging stuff, I said, number of pieces roughly doubles. I meant to say at least roughly doubles. So in the worst case, the point is that can be at least exponential.

It definitely can be more because, in general-- remember, it looks something like this-- The point is, you need at least two triangles per edge here because they need to fit together to make these little kites. So you at least double, for every edge that you visit. In the worst case, you visit the whole-- all the edges of the polygon. So you end up doubling everything.

But it can be worse because sometimes, if you don't have a lot of room in this corner, you've got to divide into lots of very tiny triangles. I think that probably only happens towards the beginning. After you've cut them small, you won't have to cut them even, even smaller. But I don't know for sure. The point is, it's at least exponential. And this is the more complicated diagram.

But I claim that you could get a pseudopolynomial bound. How do you do that? This is a little [? trickable, ?] and still have time though. So let me go over the rough idea, also what the claim is.

So pseudopolynomial bound. I'm not going to claim this for arbitrary polygons, although I think it's probably true. What we argue in the paper is that if the vertices of the polygon lie on our grid, then we're OK. It's just a little hard to keep track of otherwise. I will scale things to make this the integer grid.

And then the claim is the number of pieces is polynomial in the number of vertices, n and r-- r is usually some ratio of the longest distance to the smallest distance. In this case r is the grid size, like an r by r grid. That's like the size of the overall grid divided by the size of a grid cell. So, basically, the same thing.

So, how do we prove this? The general idea-- so we have these messy constructions, and essentially, we're inducting. We're moving one hinge, and then moving the next hinge, and moving the next hinge. And essentially, all of those inductions are nested inside each other.

You completely refine to do one thing then you have to refine to do the next one in the existing refinement. So we have a very deep recursion. It's one way to think of it. Order n depth recursion, so we end up with exponential in n.

But instead, what we can do is only recurse to constant depth. And if you're just more careful in the overall construction this is possible. How? Let me give you some of the steps.

You need more gadgets and you need to follow-- So before, I said, oh, there's some dissection out there, it's known. You triangulate, you convert triangle to square, triangle to rectangle, rectangle to rectangle, then superpose. It does the dissection, then we hinge it arbitrarily, then we fix the hinges one at a time.

Here, I want to actually follow those steps and keep it hinged dissection as much as possible. So we're going to triangulate the polygons, but in this case, we're going to subdivide further and also triangulated with all the grid points as vertices. It's little hard to draw, but here's a grid. Let's draw a polygon. Hard to make a very exciting polygon, so few vertices, but maybe something like that. OK.

If I triangulate this thing and all the interior points-- there aren't very many interior points in this example. Maybe I'll make a slightly different one. There's two interior points. I want to triangulate, with those as vertices of the triangle. So maybe I'll do something like this. A couple different shapes of triangles here, but they all have the same area. This is called Pick's theorem, special case of Pick's theorem.

So here, they're all a half-square. Even though this one spans a weird shape, it's one-half square of area. So the nice thing is if I do this in polygon a and in polygon be the triangles-- there's equal number of triangles of the same size because they have matching areas originally.

There's probably a way to do this for general polygons. I think this is the only step that requires grids except it's also a lot easier to analyze, this bound with grids. So it's, I guess, an open problem to work out without grids. OK.

The next thing is we'd really like a chain of triangles. Right now we just have a blob of triangles. And we can chainify the triangles. This is a step that was-- I don't know if I showed the figure last time. This is what we do to slenderfy everything.

We have some general hinged dissection. I don't know what it looks like. We just take each of the triangles, subdivide at their in center, cut, and then you hinge around the outside. And you'll get one-- in this case, one cycle of slender triangles. In this case, all we care about is that it's a chain.

So we have some general thing here. We subdivide each of them like this, and then you hinge around. And so now I've got a hinged collection of triangles for a, and I've got his collection of triangles for b. I'm just going to do a to b here. I should probably say that. Two shapes.

And conveniently, these triangles will still have matching areas. They're all now 1/6, if we do it right. So we get a chain of area 1/6 triangles. And I have the same number for a and for b. So this kind of cool.

Of course, the triangles could be different shapes, but I basically have a chain of various triangles. They're all the same area-- a little hard to draw-- for a. I have a similar chain for b. And I just need to convert, basically, triangle per triangle from a to b. So now my problem is a lot easier. I have these hinges which I need to preserve. That's a little trickier.

This is actually an idea suggested by Epstein before the universality result. It's like, all we need to do is do triangle to triangle while preserving two hinges. Then we could do anything to anything. So we're following that plan.

And now we're going to use all the fancy gadgets we have to do triangle to triangle while preserving these hinges and not blowing up the number of pieces too much. But definitely simpler. We're down to triangle to triangle.

Next step. OK. Next problem Yeah. This is slightly annoying. I said, oh great, these triangles are matching up. But I'm not going to be able to do triangle to triangle and get exactly the hinges I want where I want them, so I'm going to have to end up, for example, moving this hinge to another corner.

So we're going to use a new gadget, actually, for fixing which vertices connect to which triangles. This is, maybe, not obvious yet that we need this, but we will. And we're going to use a slightly, a somewhat more efficient version of, essentially, the same idea.

So we've got a hinge here, in the middle. And basically, can't control where the hinge goes, but it's supposed to go to one of the corners. So we're going to reconfigure in this way.

So we assume we have some way of doing it. And here's the thing, we assume that maybe this has already happened to a. We don't want to recurse into a because then we get exponential blow up. I'm going to have to do this for every single triangle here. There's n of them. That's a lot. I don't want to get deep recursion, I don't want to get depth n recursion.

But if I cut up in this way, in fact, I only need to cut up b. And if b hasn't been touched yet this is OK. And then I'll do it the next way, and the next triangle, next triangle, and they won't interact. That's the good news.

So how do we do it? Well, we cut up a little, oh, what do we call it, kite fan, I believe, here. Here there's two kites, and we get these triangles to match these two, these triangles to match these two. We cut up this little piece along the side. And either the green stays in here-- green is attached to the pink or magenta.

So if we keep the green in here, the triangle stays there. If we pull everything out-- and there's a little hole made here to make that more plausible. But in reality, we have to subdivide to get slender. So if we instead reconfigure the green to lie along the edge, and the blue can turn around here and fit inside because it has exactly the same shape, these two chains are identical, it can also fit in here. And then we've moved the magenta over to that side.

So that's cool. That works, and it doesn't touch a. So it's a slight variation of what we had before. And it's good. So, that's psudeopolynomial, and they don't interact. And so we can move these things however we need to according to what step four produces for us.

So this maybe slightly out of order. I could have called that step four, and this step three. Get to the more exciting part.

Finally, we do triangle to triangle. This a little crazy. I'm going to give you three constructions that give us what we want. And then I'm going to claim I can overlay them. This is what we can't do with hinged dissections, but I'm going to do it anyway. So bear with me. The final gadget will say how to overlay them.

But let's start with the relatively simple goal of triangle to rectangle. This I already showed you. And the nice thing about triangle to rectangle, this three-piece dissection, is you can hinge it here and here and it works just fine. So that's already a hinged dissection. That's the easy step.

Then we want to take that rectangle and convert it into a tiny-- or not tiny, same area, but an epsilon-height rectangle. Because remember, we have two triangles, they're different shapes so they have different heights. This one will end up being half the height, but it won't match what we'd get for this triangle. So I'm going to do steps a and b for each of the triangles. And then I have two epsilon-height rectangles.

And then the challenge is to convert one into the other. This is a challenge because they have hinges on them. So with dissections you just overlay these two cut ups. But hinged dissections, there's hinges you have to preserve, we can't do that. OK.

First part is step b, which I showed you already, going from one rectangle to another. Here's another diagram of it. It turns out it's almost hinged. You can, essentially, just flop back and forth and back and forth, except at the end you might be in trouble.

So there's one step here, and depending on parity exactly this piece of the rectangle is hinged here. But I really want to be hinged here, so I'm just going to move it over here. I have tools for moving hinges around. So it turns out, you have to check that this is safe. But you just do one hinge moving, and then you're OK.

So in this case-- this should actually go a little bit deeper-- the bottom figure shows when you go too deep you can cut, cut-- and this is just like the previous diagram of triangle to rectangle. You do that at the bottom you'll be fine. There's a couple different cases in exactly the parity and how you end up. Three cases I guess.

But in all cases the rest can be hinged. You just need this one step in the middle to fix it. So most of it is just swinging back and forth. So it's almost hinged, which is good news because we have tools to make almost hinged things actually hinged. So that's cool.

So basically, we've covered a and b at this point. But the last part is c, or how do we superpose all these things? And this is using another gadget called pseudocuts.

And essentially, you have some nice hinged dissection already, and you want to add a cut and a hinge. So just imagine cutting all the way through here and adding a hinge, I guess, on the yellow side here. And somehow, I want this thing to fold in all the ways it used to be able to fold. So it could fold into a. But then I also want it to be able to fold at this hinge, and eventually fold into b.

And it's complicated, but again, the same idea. So we've got these yellow guys, which normally live in here, and so yellows is yellow. These are triangles. These are triangles minus triangles, so they're like little quads. They have holes just the right size for the yellow. These guys have holes just the right side-- I'm sorry, how does it go?

OK. I see. It's purple, then blue, then yellow, I believe. So the yellow fits into the blue-- anyway. Whatever works. These guys nest together. And when they nest together they fill these little holes. And then there's matching patterns out here. So they all fit. How does it go?

Actually, sorry, I think they're all triangles. This just looks multicolored. So it looks like purple here is going into the cyan one at the next level. The yellow guys are going into the purple. I see. So there's a triangle and a quad here. Lovely. And then these guys stretch across.

Definitely a little more complicated. And you lose a factor of two, or whatever, but if you apply these pseudocuts in the right order-- and these are fairly simple cuttings that we have to do. We know that these cuts are mostly a striping.

So if you just apply them in order you don't get blow up. I'll just wave my hands at that. It's a little hard to draw the picture, obviously, but that's how it goes. And that's pseudopolynomial hinged dissection. This is why-- it was intentional I didn't cover it in lecture because it's pretty complicated. There wasn't time. Any questions?

Last topic is higher dimensions. Can we get a brief overview of 3D dissections? So this is more a dissection question than a hinging question, although, of course you could ask, does all this work for hinged dissections? Pseudopolynomial, we don't necessarily know.

For straight up proving that hinged dissections exist, the claim is-- it hasn't been written up formally yet-- the same techniques work. You can take any dissection and convert it into a hinged dissection. But in 3D, it turns out, dissections, by themselves, are not so simple, as a lot of open problems. Some nice things are known.

So let me tell you about 3D dissection. If I want to convert one polyhedron p into another polyhedron q, obviously, the volumes must be the same assuming we're doing a reasonable cutting and not some crazy axiom of choice thing. So volumes have to match, just like for polygons the areas have to match.

But that turns out to be not enough. And this goes back to a Hilbert problem. So you may have heard of David Hilbert. He wrote this paper of like 23 open problems at the turn the previous century, 1900. This is problem three. It wasn't directly about hinged dissections, or about dissections rather. A little bit convoluted-- it's about some certain axioms and proving certain things.

But in particular, he was asking, are there two tetrahedra of equal base and altitude, so equal volume, which can in no way be split up into congruent tetrahedra? So there's no way to dissect one into the other. If that's true it would show the certain axioms are necessary and certain proofs.

And it turns out it is true. There are tetrahedral of equal volume where you cannot do this. And that-- I don't have a slide for it-- but this was proved by a guy named Dehn. And he came up with something called the-- well, that we now call the Dehn invariant. He didn't call it that himself.

And these things must also match. It's called invariant meaning that no matter how you cut the things up and reassemble the Dehn invariant doesn't change. And so if you have any hope of going from p to q, those two things must match.

And then, Sidler-- so this was 1901, Dehn proved that this was a necessary condition. So like a year after that appeared. In 1965, a little bit later, Sidler proved that this is all that's necessary. So these are sufficient conditions. If p and q have the same volume and the same Dehn invariant, then there is actually a dissection.

And he proved it somewhat algebraically, somewhat constructively, I'm not sure exactly. There's a simpler proof by [? Jephson ?] in 1968. And he proved that, actually, in 4D the same is true. In 4D you need the volumes to match and the Dehn invariants to match, and that's enough. In 5D and higher no one knows what it takes for a dissection. Pretty weird. It could be interesting to study these more carefully.

Let me tell you briefly about Dehn invariants. A little awkward unless you're familiar with tensor product space. How many people know about tensor product space? A few. OK. If you've done quantum stuff, I guess, it's more common. I'm not familiar with tensor product space, but here we go.

Tensor product space. But I can read Wikipedia with the best of them. It's a fairly simple notion, it just has somewhat weird notation. You can do things like take something x and write tensor product with y. And what this means is, basically, don't mess with this product. OK. It's a product.

Really, this is two things, x and y. They're not interchangeable, they're in completely different worlds, different units, whatever. You can't like multiply them. They just hang out side by side. You also can't flip them around. It's not commutative. OK. Fine.

But some things hold. Like if you take, I don't know, z and add it to this product, you do have distributivity, so you can get x-- is this-- no, this doesn't look very correct. If I have this then you can multiply that out. So you get x tensored with y plus x tensored z. So that holds. It also holds on the left.

And the other thing is that constants come out. So if we have c times x tensored with y, this is the same thing as c times x tensored with y. So in the end I'm going to have a bunch of these pairs, these tensor pairs. And I'm also able to add them together. And nothing happens when you add them together, they just hang out. So in general-- you could also have a constant factor-- so you have a linear combination of pairs, basically.

Why am I doing this? Because here's the Dehn invariant. Dehn invariant says, look, with polyhedra you've got two things-- it's going to be the x and the y over there-- you've got edge links and you've got dihedral angles. So look at every edge.

Here's an edge of my polyhedron here. It has some length, which I'll call l of e. And there's some angle here, which I'll call theta of e. Add those up over every edge.

So the Dehn invariant is going to be the sum over all edges of the length tensored with the angle.

AUDIENCE: Isn't an angle a function of two [? of these? ?]

PROFESSOR: Angle is the angle between these two planes. So that's a dihedral angle. Yep. So for every edge there's one dihedral angle. Just sort of the interiors of all that angle there at the edge.

So this is kind of what's going on. And these things have to match. Now it's a little more complicated.

Sorry, it's not really just the angle. Essentially, if you add rational multiples of pi nothing happens. So you actually take this weird group, all rationals times pi-- All this means is if you have two angles and their difference, that you subtract them and you get a rational multiple of pi, then those two angles are considered the same. So what this is really saying is I only care about the irrational part of pi, roughly. You add pi over 2, that doesn't change anything.

Why this thing? Well if I take an edge, and for example, I cut it in half anywhere, I could cut it at an irrational fraction, or whatever, I will get two lengths but they'll be tensored with the same angle. I didn't change the angle. And so by distributivity-- once you get things inside the same place.

So in this case, we'll get two lengths that add up. They match. OK. So as long as you have matching angles you can add the lengths together. That's what distributivity tells you.

Similarly, if I tried to cut this angle in some piece, it could be an irrational ratio between the two pieces, they will have the same edge length. And when I have matching edge lengths I can use distributivity and add the angles back together. So basically, when you dissect, this thing will not change.

It's a little more awkward when I cut here because this was originally a pie, and then I cut it into some pieces. And this is where you need the rational multiples of pi not mattering. But eventually you can prove Dehn invariant is invariant.

The harder proof, you can prove that it's also sufficient if you have the matching volumes. As recently proved like a few years ago, 2008, that whether the Dehn invariant of one polyhedron and another match is decidable. So there is an algorithm to tell whether two polyhedron have the same this thing.

Decidable is a pretty weak statement. Natural open problem is, is there a good algorithm to do it? We don't know. If it does match, is there a good algorithm to find the dissection? We don't know.

These may be easy if you really understand the proofs deeply. But at the time no one cared about algorithms. At this point, we need to go back and really understand how to actually do 3D dissections so that we could then do a 3D hinged dissections. That's it.

Don't forget, orgami convention is on Saturday. Should be fun.