

PROFESSOR: All right, lecture 19 is about mostly refolding, common unfoldings of polyhedra, convex polyhedra, and also about Mozartkugel. But most questions were about the common unfolding stuff, so let me start with that.

First question-- is this fractal unfolding of regular tetrahedron and the cube, which looks like this-- is it resolved? The answer is no, but it is at least published now. This is at EuroCG. One of the authors is here, Ryuhei Uehara. And it's still conjecture, although there is a way to explicitly construct this. This does actually fold into two things. It folds into a cube and a tetramonohedron, a tetrahedron with equal sides. It's not a regular tetrahedron, though, but this is actually an unfolding.

And they have another one, which they haven't drawn because it has 10^{180} edges or so, but it is within 3 times 10^{1796} of regular. So that's pretty good evidence that this is gonna work out. Let me write that down. So they have a particular iteration procedure. It's not always guaranteed to make a connected unfolding, is the troubling part. I think it will converge, but the challenge is to find an infinite sequence of examples like this one, that form a connected polygon, that would limit to the fractal.

They have that instance. They have another instance which is very, very good. This is the error. But we don't yet know whether there are infinitely many examples the converge all the way to some fractal set. So that is a state of the fractal.

Next we have a question about boxes. When that lecture was given, it was open whether there was a common unfolding of three boxes. That problem is now solved.

There's two papers about this topic. The first one we wrote together. And this is just a fun example, not yet related to the three box problem, but you may recall there were some unfoldings of two boxes where one was 45 degrees to the other. This was the challenge of making it not 45 degrees, something else. And here it's a 3 by 1 triangle that is the axis. So you end up with $\sqrt{10}$, $\sqrt{3^2 + 1}$ squared. So that's still two boxes, but at least at a funny angle.

And this was actually found computationally, with an integer linear program for this. Integer linear programming-- set up the problem of which pixels are in and which pixels are out in this representation, and could you fit-- we started with these parameters. You have to be very careful about how you choose the numbers to make the surface areas match. So we just chose that by hand, and then tried to find whether there was such a thing. You could formulate that as an integer linear program. It's not guaranteed to be solvable especially efficiently, but this one we found a solution.

For fun we also turned it into a puzzle, where we were just modified the boundaries in corresponding ways, so now finding either folding is actually pretty challenging. Well, one of them's probably easy, but the off center one is challenging.

But this paper also had a first kind of solution to the three box problem. So this is one net-- it's a grid unfolding, even-- and it folds into 1 by 1 by 5, 1 by 2 by 3, and 0 by 1 by 11. So this is a flat box, a doubly covered rectangle. It's also not technically a grid unfolding-- we're cutting in the middle of a pixel, here. But if you just refine by 2 and call this 0 by 2 by 22, 2 by 4 by 6 and then 2 by 2 by 10, then this is a grid unfolding. I don't know if you care whether it's a grid unfolding, but you can make it one if you want.

So this is the first solution, and this is actually found by computer search. This came from a long series of research in the two box problem. We started by looking at all common unfoldings of 1 by 1 by 5, 1 by 2 by 3, and then saw whether any of them folded in to 0 by one by 11. So the just for fun, the way that search was done-- which is in the same paper-- is essentially to try to construct incrementally a common unfolding of those two polygons so they both have surface area 22.

The idea is, you start with a single pixel and you say, oh, a single pixel can fit on both surfaces. Then you try all possible ways to add a second pixel, and make sure that at all times the thing you have is a partial unfolding of both shapes. So that way you get to throw away anything that's guaranteed to be bad, that won't fit on either shape. And so as you grow your polygon, you get more and more options. So the

max we get to here is about 5 million common partial unfoldings of size 17. And for comparison, the total number of ways to build polyominals of size 17-- this is only looking at grid unfoldings-- so all possible polyominals of size 17 is 10 times that. So we're getting a lot of savings in time by throwing away things that are not common partial unfoldings.

And then, conveniently, things start getting smaller. You get to throw away more and more stuff, and you end up with only 2,263 common unfoldings of these two shapes, top. And surprisingly, exactly one of them folds into a 0 by 1 by 11 box. So this is unique for these three parameters. And unfortunately that's kind of the limit of how far we could easily go with a computer search. And we could maybe get a little bit bigger, but it seemed like we were not going to be able to find three boxes just by computer search if we wanted a non-flat example, because numbers are just going to have to get bigger, and we're not to be able to do this enumeration procedure.

So we need to think, instead of computer search. Before we get to that, one other fun result we got from thinking is, if you allow these flat doubly covered rectangles, there's one polygon, namely a long strip, that will fold into arbitrarily many distinct rectangles. So if you take a long strip, you can do this zigzag pattern. In this case it's height 2, and you alternate covered on top, covered on the bottom, covered on top, covered on the bottom, and you turn around and do the same. And if provided your dimensions are properly divisible, this will make a doubly covered rectangle of this size. And you could also do it with-- this is height 1, height 2, height 3, and so on.

In general , if you have and l by one strip you can make about l different rectangles, depending on how you choose your angle to be one of some integer thing. So that's cool. If you allow doubly covered things, then you can actually make many, many boxes.

If you really want 3D boxes, that's where the next paper comes in. So next paper is about three different orthogonal boxes, all not flat. It's by Shirakawa and Uehara again. And here's one of the solutions. So this one polygon is a grid unfolding-- folds

into a 7 by 8 by 14, a 2 by 4 by 43, and a 2 by 13 by 16. And this is almost certainly beyond what we could ever do by computer search, by just exhaustion. This is found using human intelligence, and this grid pattern.

And they have another example. This one falls into 2 by 13 by 58, 7 by 14 by 38, and 7 by 56. This is actually-- this a little more complicated, or a little bigger, I should say. They're both based on a common principle, which is to start from a solution to the two box problem and then modify it to make three boxes. So this is a common unfolding event a by b by $8a$ box, and an a by two a by $2a + 3b$ box for, I think, for any values of a and b. And essentially the way you see that is, you either wrap it around this way, or you wrap it around this way. I think that's how the two foldings work. You have to figure out which of these lines to keep, and which ones to throw. I haven't color coded them here.

But that's a general thing that's known before. I think maybe we even saw in lecture this-- there was an example where you could just repeat, and get infinitely many examples of two different boxes. Now we want to modify it to make three different boxes, or to add a third box. It's not known how to do a fourth box of nonzero volume.

So the basic idea, which doesn't work, is following-- if I built some box, and I slit it in this sort of h pattern, and then I refold it to be like this, I should get a different box. That would be really cool. The problem is, if you look at what lengths have to be equal for this. So these two lengths end up being these two lengths, so those sums must be equal. And over here that same length must equal the two heights of the boxes. That implies that actually this length equals this length, and so this is actually a square. It's drawn here magically to not be squares, but in fact they must be squares, so it's good example of misleading figure.

Think about it as like, oh, man, they have to be squares. That means you're converting a 2 by 1 rectangle to a 1 by 2 rectangle. And that's really not going to change anything. You just rotated your box, which is frustrating. Fortunately, there's a tweak to make it work.

The tweak is, you add wiggly tabs and you refold things. So this is a-- in terms of the grid, here, we've got an 8 by 7 rectangle originally on this face. We do all these cuts that are in bold, and then when we refold we also add in these dash creases. That's kind of part of the magic. And we end up with a 2 by 13 rectangle. These don't even have the same surface area. Some of the material got moved into the adjacent sides. But if you stare at it long enough, this is a valid transformation of an existing box.

Now, of course, the challenge is to get this trick to work with this unfolding of two boxes. And it'll be really nice if you could make it work for both the boxes, then you'd get four boxes, total. We don't know how to do that. But with this particular augmentation-- so, same picture, particular values of a and b chosen here, but these extra tabs and pockets are put in there. Then we get a common unfolding of three boxes. And you can see from the numbers here you've got the 2 by 13 and the 7 by 8 appearing, And this comes from the other folding.

So, pretty cool, right? And the same gadget is being used in this the first example I showed you, which is a somewhat smaller one. Again you get 7 by 8 here, and 2 by 13.

So, this is how we know how to do boxes. Now, in fact, there are infinitely many examples along these lines. You can change that grid. Parities have to be right and maintain some proportions, but there are lots of examples of taking that grid, applying it to this net, and then you get these three boxes. For any integer, k greater or equal to zero will work. And so you get a nice infinite series of examples, I think.

The rough ratios are almost the same, but-- is that right? No, no, actually they're changing the proportions. You've got k , $4k$, $16k$, versus $4k$, $4k$, $7k$, versus $4k$, k , $4k$. So-- sorry? Oh, $4k$, $2k$, and $24k$, sorry-- and this is 32. Yeah, sorry. I need to expand on that that. So, pretty cool, and this is the state of the art of common unfoldings.

Next open problem is for non-zero volume boxes. Can you prove that 100 nonzero

volume boxes are impossible? Who knows. Maybe even four or five is impossible. It seems pretty tricky.

OK, I have one more question, which is about the smooth unfolding stuff. I call this a limit, and there's this issue about the area. So I want to go into this. Remember this picture-- we take a smooth prismatic, which is two convex-- not polygons, but two convex bodies, parallel. Take the convex hull. You get the smooth surface. And then we unfold it in the volcano way of taking every rib here, every rule line, and developing it out flat. And when you do that, you take the union of the ribs and the unfolding, does not have the same area as the original surface area of the polyhedron, which seems a little weird. Sometimes we might call I might have called this a limit. It's not really a limit of the discrete case. And this is why we don't know how to solve the discrete case, and yet, we can solve the smooth case.

The issue is, if you wanted to form this structure as a limit of discrete prisms, you want to subdivide this into a very fine polygon, subdivide this into a very fine polygon, and then try to unfold those discretely, and then take the limit as you take finer and finer, closer and closer approximations of the smooth things as polygons. It really depends how you discretize-- maybe one way to think about that is, if I start here and I do uniform spacing from this point-- or I do it from this, point or from this point-- each of those is going to have a different-- it's going to capture a somewhat different discrete structure. I think the way to think about that is, the gaps will be spread out in different ways. If I always cut here, there will always be a gap incident to that point, whereas if I always cut here, there will always be a gap incident to this point, and not necessarily at this point.

Depending on your discrete approximation, you'll get a very different unfolded diagram. You're going to get holes out here, because when you do the discrete thing, the discrete thing does preserve area. And yet, you imagine, as you take the limit somehow the area jumps at the end. That shouldn't happen for a limit, so this is not a limit. What happens is, you have many different limit points depending on your discrete representation. And this green thing is essentially the union of all those limit points. When you take the union, the area goes up. That's essentially what's going

on here. That's my best interpretation. But yeah, shouldn't use the word limit.

OK that's it for questions, unless there are other questions about this or anything we talked about in lecture.

So the next topic I want to talk about-- this is a bonus, extra topic hasn't been covered in this class before, but I've always wanted to. I think there was a final project about it once. It's this guy, Theo Jansen, and he builds these crazy linkage structures. And the next two classes are about protein folding, which is more on the linkage folding aspect, and so thought it'd be fun to transition over. Maybe first I'll just show you some.

These are two kits you can buy of his constructions. This guy-- I will blow.

[BLOWS]

This guy's on here. If I turn this guy, he walks-- and walks pretty elegantly if I turn this very uniformly. It's pretty cool. It can walk both directions. It's also unpowered. You blow it, and it works. Here's another example. You see just the legs. It's more clearly a linkage, although this is actually just implementing the same linkage. It's just got some thicker-- filled in some rigid triangles here.

[BLOWS]

This is fun, because the wind's going in one direction, and the guy's going in the opposite direction. Here it's going in the direction of the wind. Here it's going perpendicular to the wind.

[BLOWS]

A little bit slower, though. I'll just manually turn this guy. Is he going a little faster? So it's got really cool leg mechanism. Looks almost like an animal. Theo calls these his children, his animals. His goal is for them to be self sufficient and to live on their own in the beach. How many people have seen his TED Talk? A few people. You should watch it. I'm not going to show it here, because some people have seen it.

He has this vision. He makes them all out of PVC. These are just injection molded plastic kits which you can buy. They're super cool. Let me show you the bigger ones. This is living on the beach. At this point they're not self sufficient, he has to correct them occasionally. . But this guy's just walking in the direction of the wind. I think here the wind slows down a little bit, so lose momentum, then picks up.

He does this in the Netherlands. He's got this crazy beach. Here's a much bigger one. It's got a lot more wind collectors at the top. So these are all self powered. Originally he pushed them along, but now they've learned to move by themselves. And very cool legs-- I want to talk more about the leg linkage, because that's the more mathematical part.

So this is his original drawing of the leg. It's a pretty simple idea. It's a pretty simple linkage. You've got a quadrilateral here, 4 bar linkage, C-K-J-B, and you've got a rigid triangle attached here. Then another quadrilateral here, and then a rigid triangle at the bottom. So it's really just two quadrilaterals connected by two rigid triangles. And so it's a one degree of freedom mechanism. As you flex this quad, this one has to adapt because that quad controls all three of these points because of the rigid triangles, and so the fourth one is determined. And then the crank is here. So this says vaste printer, which means fixed points. So you've got these two fixed points-- this is actually also fixed. This is just an x to y ratio. So these two guys are pinned, and this guy is rotating around this fixed point-- that's the actuation, and that causes this quadrilateral to fold. We are basically setting the center of rotation of this guy relative to this guy by measuring these lengths, a and l.

He has these numbers, which he calls the 11 holy numbers. It's a little cause because there's actually 13 of them. There are 11 edge lengths, and then there's these two, which are not really edge lengths. It's just a measurement of how to put the center of rotation relative to this guy. And the reason why x and y matter here is because there's actually a floor at the bottom, and you have to orient yourself correctly relative to the floor so you can actually walk.

So he found these numbers by genetic algorithm. In the early days there weren't-- I

guess, before there were a lot of genetic algorithms around. So he tried lots of different values, simulated them on the computer, found which ones walked best, basically. Ones that were walking well in various kinds of terrain would survive to the next generation, try various random mutations, perturbing all of these numbers, and eventually finding this set to be especially good in simulation, and that's what all of the built linkages are based on.

This is his computer drawing, plus some markings on top of it. If you take this linkage and go through the one degree of freedom, this point, of course, moves along a circle. Then we're just drawing lots of instances on top of each other. The foot-- this guy down here moves in this pattern, which is pretty cool. This is basically lifting your foot up off the ground, pushing it down, and then dragging along the floor. That's how you walk, if you think about it. You push and propel yourself forward. So this scraping along the floor, if you've got a high friction foot, will actually move you forward.

In some sense, all of this design-- and the one way you would choose these numbers-- is to guarantee that A- you get lift, because if you have non-uniform terrain, you want to walk. He likes to call this mechanism better than the wheel. It's like every invention of something like the wheel, but even better because wheels don't really work if you have steps on the ground. Walking is superior, cause you could walk over bumps and steps and things like that. So you want a good amount of lift, and then you want a good amount, and ideally-- you can kind of see this here. These are uniformly spaced as you rotate around, and these are all-- they look uniformly spaced on the bottom, which means you're applying uniform force all along the floor contact. It's not uniform here, but it's uniform down here, mostly. It's not actually straight in the true mechanism.

I think this is another drawing, two other people's drawings of the same mechanism. Here the triangles have been filled in to be-- I think this is a design for a metal version. And then this is a mathematical simulation. Again, uniformly spaced rotary cranking here. You get to see the spacing of the points here. So it's moving a little bit faster over the lift, and then pushing down and spending a fair amount of time

here, and pretty uniformly applying . force. Also not completely flat. It looks like it's convex-- in fact there's a little upward dip here. But it's pretty good, pretty straight, and a really good walking mechanism, possibly the best within this family. Depends how you define best, of course.

And here's a fun animation of how the parts of the linkage and adding multiple legs, which is I wanted to show it. So you start with one 4 bar mechanism. These are probably not quite the right lengths, but close. You add two rigid triangles it's a little add another quadrilateral on there you see the spitting part offset relative to that and then here you get the trajectory of the foot. And you see on a terrain how it would move you forward, especially if you have multiple legs, then one of them is always in contact so you're moving forward. And then if you want to be even cooler, you go to three dimensions, use even more legs. And you have a little crank in the center that offsets the timing of the feet so they're all sporadically moving, and that's essentially what's going on in this structure where they're nicely, evenly distributed. There's a central crank here that just offsets the timing of all these feet, and it's super cool. I mean, some ways the coolest part is how it looks like a real animal walking-- the funny timing, and the way you lift and then spend most your time with a foot on the ground, and so on. So, fun stuff. Let's see.

Of course, other people have tried to improve it in various ways. There's a couple of papers proposing alternate walking linkage mechanisms. This is the Theo Jansen version, this is another version by Ghassaei. And it's actually a simpler linkage, just a 4 bar linkage over here too rigid triangles and another 4 bar linkage. I guess roughly the same number of parts but, I think, one fewer edge if I counted correctly, just because of the way they're combined.

And some fun things as you get-- this is the foot pattern down here, still has a nice straight part, looks pretty uniform, speeds up for the lift. Does not lift as high-- this one has a 30 percent higher step, which is considered better. You could move over rougher terrain. This one has other Jansen's has other advantages, like if you have two of them in the same plane they will not collide with each other. Two legs reflected around each other. This one, they would collide. You have to move them

to separate planes. But some fun things here as you get that this foot pattern is symmetric, down the y-axis, and you get much less variation in the height, sort of more uniform, and a little less velocity variation on the stride part, the walking.

And then the other thing, which I think was the main purpose of the design, was that the center of mass of the whole structure doesn't move very much. The blue dots here are the center of mass over the motion of the Jansen mechanism. And this one, the blue dots are much smaller areas, 85% less movement. I'm not sure of the exact definition of 85% there. This should be, if you were like riding on the thing, you should feel less wobble, in principle. They did build one of these out of PVC tubing as well, but I haven't seen any video so I don't know how it looks. Be kind of neat to see

So this is a two fat example, apparently. Said this one was over eating, and so it tends to fall over. So it's a challenge to get these things to work and you-- I can't imagine how many hours he spends building these, and then for one of them to not live. He does things like, he'll run these experiments. Like genetic algorithms is one thing, but he'll do the real genetics thing of building lots of different variations, put them on the sand, see which one goes the fastest. The other ones he kills. The one who went was the fastest survives to the next generation. But each experiment must take hours and hours-- really impressive.

So there are issues in getting these things to work, and just having the leg mechanism is one thing. These guys have a lot of very interesting mechanisms beyond the legs, in order to-- they're sensors, essentially. They're purely mechanical-- there's no electronics here that are physical, I should say. There are sensors to detect things like the wind direction.

Recently we had the honor of co-exhibiting with Theo Jansen. So this is some of our curve crease paper folding in this, or some of his structures. Some of these small kits and photographs of his larger ones were at this show in Stonybrook, New York, earlier this year. It's kind of fun.

And while we're on the topic of kinetic sculpture, I have to show Arthur Ganson. It's

funny, Ganson and Jensen are so close. This is our local kinetic expert-- kinetic sculpture expert. These are four examples I took from the web, but all of these examples and many, many more at the MIT Museum. If you haven't seen his exhibit, you must. He hand builds lots of gears and cranks, and uses them to build crazy things. This is called the self-oiling machine. it just grabbed a bunch of oil and it' about to pour all that oil onto itself. It's really gross. There it goes, pour-- and should never get rusty. This is a very simple mechanism that's just the random and cool how the changes kind of move, wiggles around. It's a force to be planar, essentially because of that chain.

This one-- you've got various handmade gears and so on, and hand cut pieces of rough paper, to make them fly. And this is another beautiful, relatively simple one. You've got a big counterweight to make this chair almost zero weight, and then there's a little cat here that's just rigid and going back and forth and it causes that chair to dance around in very cool ways.

It's pretty amazing. He lives locally, and has made a huge array of sculptures, one which I don't have a video of, which is a lot of fun, is just-- there's a big crank, and then a series of like 20 to 1 gear reductions, and at the other end the turning element is in a solid block of concrete. And there's some computation, like it'll take 100,000 years before that before it turns that concrete around, which it can't do. Initially the gears are going pretty fast, and then, slower and slower, and then you just can't see them for like five iterations and then it's wedged into concrete It's a fun, practical example-- practical Anyway, lots of very cool kinetic sculpture and all somehow based around linkages although not necessarily strictly following our definition. And next week we'll be talking about protein folding.