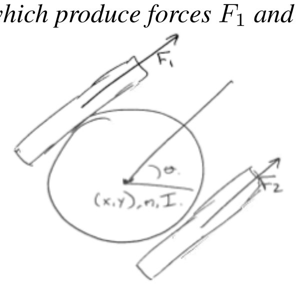
This problem set is due *by 11:59pm* on **Tuesday, Feb 17**.

Problem 1 (Definition of underactuated) *For each of the following systems, decide whether each control problem is fully-actuated (in all states), or if there are any states in which the problem is underactuated. Use the definition of underactuated provided in lecture. Explain your answers.*

a) ***(1 point)*** *A planar submarine with three propellors (arranged as below, thrust axes are 30 deg away from each other, and the center of mass is 2m in front of the intersection of the three axes). Assume that the propellors can produce a thrust both forward and backward. The task is to regulate the position (x, y) and orientation (θ) of the submarine.*

b) ***(1 point)*** *Consider the standard two-wheel "trashcan" robot sketched below. The configuration of the robot is given by $[x, y, \theta]$, and control system applies torques at the wheels which produce forces F_1 and F_2.*



c) ***(1 point)*** Holonomic *constraints are equality constraints that can be expressed purely in terms of the configuration (position) of the system.* Nonholonomic *constraints are "nonintegrable" constraints on velocity (e.g., the system* can *get to any configuration, but cannot get there by any arbitrary path). The system in part (b) is a classical example of a nonholonomic system, which is subject to the constraint:*

$$\dot{y} \cos \theta - \dot{x} \sin \theta = 0.$$

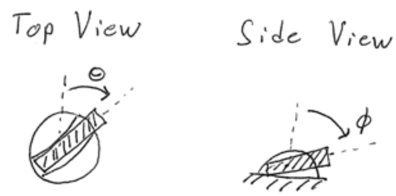
If nonholonomic systems have constraints on velocity, then are all nonholonomic systems also underactuated? Explain your answer.

d) ***(1 point)*** *A telescope is aimed at the sky using the system sketched below. Assume the dynamics are given by*

$$\left(I_h + \frac{ml^2}{12}\sin^2(\phi)\right)\ddot{\theta} + \frac{ml^2}{6}\sin(\phi)\cos(\phi)\dot{\phi}\dot{\theta} = \tau_\theta$$

$$\ddot{\phi} - \sin(\phi)\cos(\phi)\dot{\theta}^2 = \frac{12}{ml^2}\tau_\phi$$

where you can control τ_θ and τ_ϕ. Is the system ever underactuated if the outputs one wishes to regulate are ϕ and θ themselves? What if you wish to control the point (x, y)[1] *on the sky towards which the telescope points? Explain.*



Problem 2 (The simple pendulum) *We will now investigate some aspects of the dynamics and control of the simple pendulum, whose equation of motion can be written:*

$$ml^2\ddot{\theta} + b\dot{\theta} + mgl\sin(\theta) = u.$$

a) ***(3 points)*** *For this part, we will consider the full dynamics of the simple pendulum with a constant input torque, but ignore any wrap-around effects. We wish to numerically investigate the basin of attraction of the stable fixed point for three parameter sets: $\{\{b = u = 0\}; \{b = 0.5, u = 0\}; \{b = 0.5, u = \frac{g}{2l}\}\}$. For each setting of the parameters, give the location of the stable fixed point and a plot of its basin of attraction over the domain $x \in \{-2\pi, 2\pi\}$ and range $\dot{x} \in \{-2\frac{g}{l}, 2\frac{g}{l}\}$. Use $m = 1$, $l = 1$, $g = 9.8$, and submit separate plots for each parameter set.*

Hint*: A matlab routine containing the basic components to compute and plot the basins of attraction for the simple pendulum is available on the course website (`calc_basin.m`). The existing code creates a mesh over the state space which you can use to keep track of what states are in the basin.*

b) ***(3 points)*** *Using the same pendulum parameters as for the previous part, plot the phase space trajectory of the pendulum for $b = 0.5, u = 0$ from the intial condition $\theta = \pi/4$, $\dot{\theta} = 0$. Use feedback linearization to eliminate damping on the system, and plot this phase space trajectory.*

c) ***(1 point)*** *If you use feedback linearization to double gravity for the now effectively undamped system, how will it change the undamped phase plot you found in the last part? In addition to the torque used to cancel damping, how much more torque must your motor be able to output to double gravity? To invert gravity?*

[1]If you wish to be pedantic, project the visible sky onto a plane, accepting the distortion, and denote the point the telescope is viewing on this plane by (x, y).

Problem 3 (Optimal control of the double integrator) *In this question, we will look at controlling the double integrator system (i.e., "the brick"). For all parts, assume the brick has unit mass, giving the equation of motion as follows:*

$$\ddot{x} = u.$$

Hint: *A matlab routine containing the basic framework for implementing the required controllers for this problem is available on the course website (*`brick_control.m`*).*

a) *(**2 points**) In class we used Pontrayagin to determine the optimal* minimum time *policy to get to $(0, 0)$ for the brick. In Matlab, encode this minimum-time policy when $|u| \leq 1$. Give the phase space trajectory of a brick following this trajectory from the initial condition $x(0) = 2$, $\dot{x}(0) = 1$.*

b) *(**3 points**) Using* `lqr` *in Matlab, determine the infinite-horizon LQR policy for the brick when $\mathbf{Q} = .25\mathbf{I}$, where $\mathbf{I}$ is the identity matrix, and $\mathbf{R} = 10\mathbf{I}$. Plot the phase space trajectory of this policy from the same initial condition as in the previous part, and compare the trajectories the system takes when following the two different policies.*

c) *(**2 points**) Compare the time it takes the minimum time policy and the LQR policy to get within .05 of the goal in both x and $\dot{x}$. How does the time for the LQR policy change as Q is increased? When Q is set to 100 times the identity matrix and R is kept the same, how does the time taken with LQR compare to time taken by the min-time policy? Does this make sense?*