Lumped-Element System Dynamics

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***(with thanks to SDS)**

Outline

- Our progress so far
- Formulating state equations
- Quasistatic analysis
- Large-signal analysis
- Small-signal analysis
- Addendum: Review of 2nd-order system dynamics

Our progress so far…

- Our goal has been to model multi-domain systems
- We first learned to create lumped models for each domain
- Then we figured out how to move energy between domains
- Now we want to see how the multi-domain system behaves over time (or frequency)

Our progress so far…

- **The Northeastern/ADI RF Switch**
- **We first lumped the mechanical domain**

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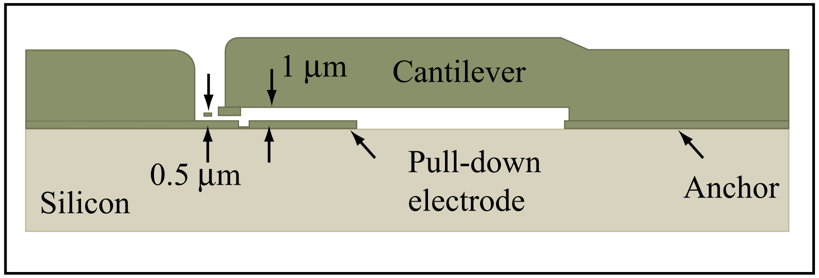
Figure 11 on p. 342 in: Zavracky, P. M., N. E. McGruer, R. H. Morrison, and D. Potter. "Microswitches and Microrelays with a View Toward Microwave Applications." International Journal of RF and Microwave Comput-Aided Engineering 9, no. 4 (1999): 338-347.



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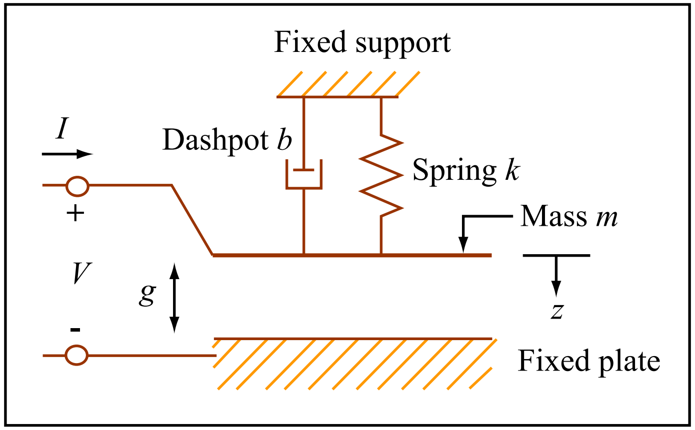
Adapted from Rebeiz, Gabriel M. *RF MEMS: Theory, Design, and Technology*. Hoboken, NJ: John Wiley, 2003. ISBN: 9780471201694.



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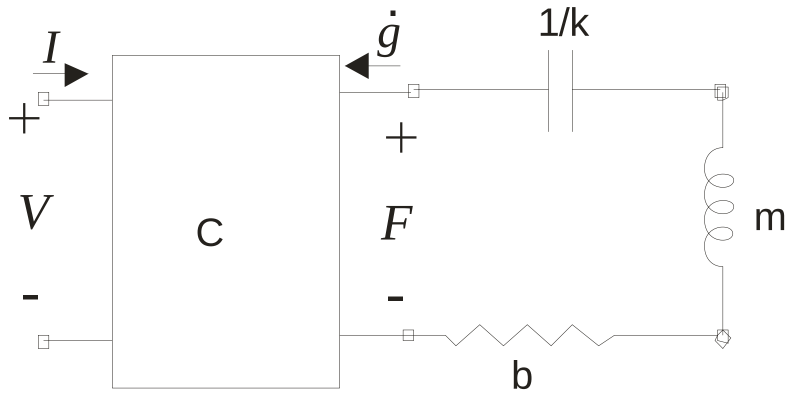
Adapted from Figure 6.9 in Senturia, Stephen D. *Microsystem Design*. Boston, MA: Kluwer Academic Publishers, 2001, p. 138. ISBN: 9780792372462.

Our progress so far…

> **Then we introduced a two-port capacitor to convert energy between domains**

- **Capacitor because it stores potential energy**
- **Two ports because there are two ways to store energy**
  - » **Mechanical: Move plates (with charge on plates)**
  - » **Electrical: Add charge (with plates apart)**
- **The system is conservative: system energy only depends on state variables**

$$W(Q,g)=\frac{Q^2 g}{2\varepsilon A}$$



Our progress so far…

- We first analyzed system quasistatically
- Saw that there is VERY different behavior depending on whether
  - Charge is controlled
  - ➔ stable behavior at all gaps
  - Voltage is controlled
  - ➔ pull-in at $g=2/3g_0$
- Use of energy or co-energy depends on what is controlled
  - Simplifies math

$$F = \left.\frac{\partial W(Q,g)}{\partial g}\right|_Q = \frac{Q^2}{2\varepsilon A}$$

$$V = \left.\frac{\partial W(Q,g)}{\partial Q}\right|_g = \frac{Qg}{\varepsilon A}$$

$$dW = VdQ + Fdg$$

$$dW^* = QdV - Fdg$$

$$Q = \left.\frac{\partial W^*}{\partial V}\right|_g = \frac{\varepsilon A}{g} V_{in}$$

$$F = \left.\frac{\partial W^*}{\partial g}\right|_V = \frac{\varepsilon A V_{in}^2}{2g^2}$$

Today's goal

- How to move from quasi-static to dynamic analysis
- Specific questions:
  - How fast will RF switch close?
- General questions:
  - How do we model the dynamics of non-linear systems?
  - How are mechanical dynamics affected by electrical domain?
- What are the different ways to get from model to answer?

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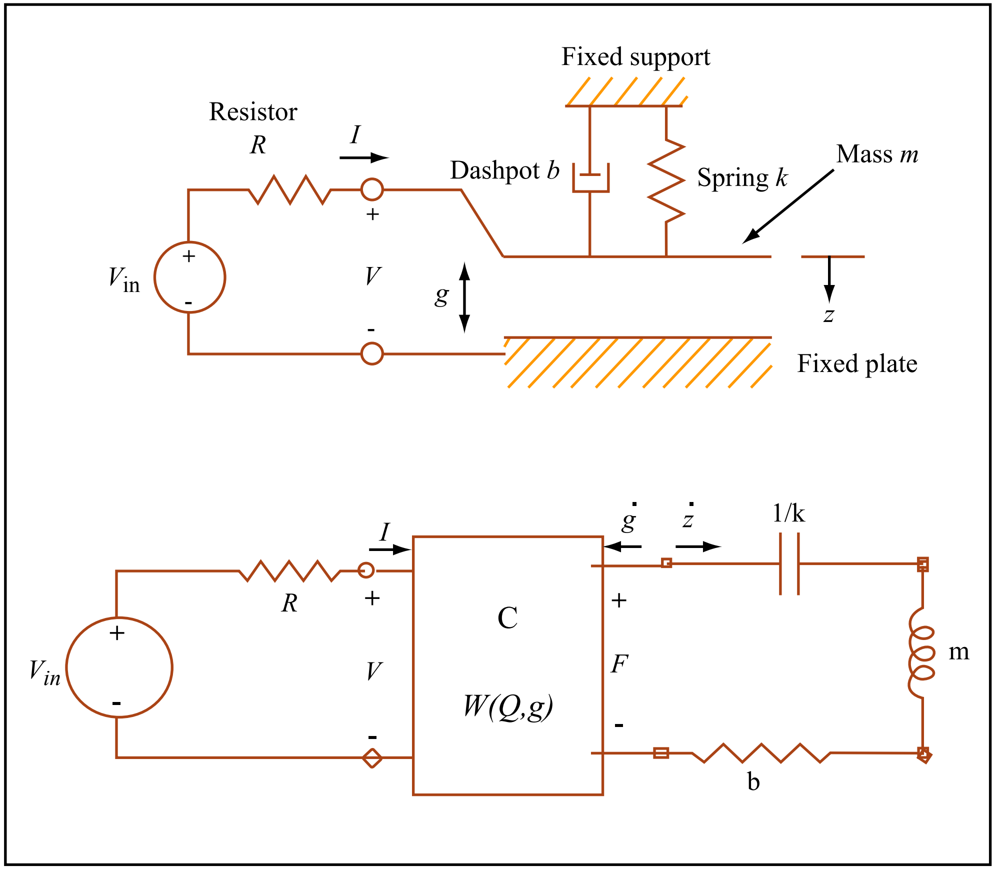
Adding dynamics

- Add components to complete the system:
  - Source resistor for the voltage source
  - Inertial mass, dashpot
- This is now our RF switch!
- System is nonlinear, so we can't use Laplace to get transfer functions
- Instead, model with state equations



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Electrical domain | Mechanical domain

State Equations

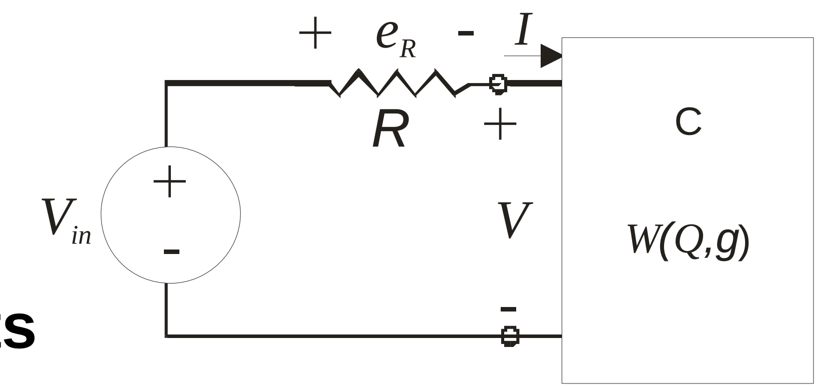
- Dynamic equations for general system (linear or nonlinear) can be formulated by solving equivalent circuit
- In general, there is one state variable for each independent energy-storage element (port)
- Good choices for state variables: the charge on a capacitor (displacement) and the current in an inductor (momentum)
- For electrostatic transducer, need three state variables
  - Two for transducer (Q,g)
  - One for mass (dg/dt)

Goal:

$$\frac{d}{dt}\begin{bmatrix} Q \\ g \\ \dot{g} \end{bmatrix} = \begin{pmatrix} \text{functions of} \\ Q, g, \dot{g} \text{ or constants} \end{pmatrix}$$

Formulating state equations

- Start with Q
- We know that $dQ/dt=I$
- Find relation between I and state variables and constants



$$\text{KVL}: V_{in} - e_R - V = 0$$

$$e_R = IR$$

$$V_{in} - IR - V = 0$$

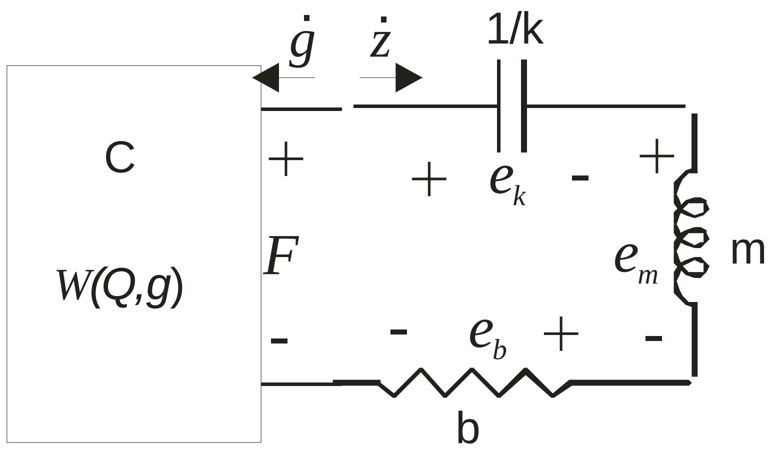
$$\frac{dQ}{dt} = I = \frac{1}{R}\left(V_{in} - V\right)$$

$$V = \frac{Qg}{\varepsilon A}$$

$$\boxed{\frac{dQ}{dt} = \frac{1}{R}\left(V_{in} - \frac{Qg}{\varepsilon A}\right)}$$

Formulating state equations

- Now we'll do $\dot{g}$
- We know that $d\dot{g}/dt = \ddot{g}$



$$\text{KVL}:$$
$$F - e_k - e_m - e_b = 0$$

$$e_k = kz$$
$$e_m = m\ddot{z}$$
$$e_b = b\dot{z}$$

$$F - kz - m\ddot{z} - b\dot{z} = 0$$

$$z = g_0 - g \Rightarrow \dot{z} = -\dot{g}, \ddot{z} = -\ddot{g}$$

$$F - k(g_0 - g) + m\ddot{g} + b\dot{g} = 0$$

$$\ddot{g} = -\frac{1}{m}\left[F - k(g_0 - g) + b\dot{g}\right]$$

$$\frac{d\dot{g}}{dt} = -\frac{1}{m}\left[\frac{Q^2}{2\varepsilon A} - k(g_0 - g) + b\dot{g}\right]$$

Formulating state equations

- State equation for g is easy: $\frac{dg}{dt} = \dot{g}$
- Collect all three nonlinear state equations

$$\frac{d}{dt}\begin{bmatrix} Q \\ g \\ \dot{g} \end{bmatrix} = \begin{bmatrix} \frac{1}{R}\left(V_{in} - \frac{Qg}{\varepsilon A}\right) \\ \dot{g} \\ -\frac{1}{m}\left[\frac{Q^2}{2\varepsilon A} - k(g_0 - g) + b\dot{g}\right] \end{bmatrix}$$

- Now we are ready to simulate dynamics

Outline

> Our progress so far

> Formulating state equations

> Quasistatic analysis

> Large-signal analysis

> Small-signal analysis

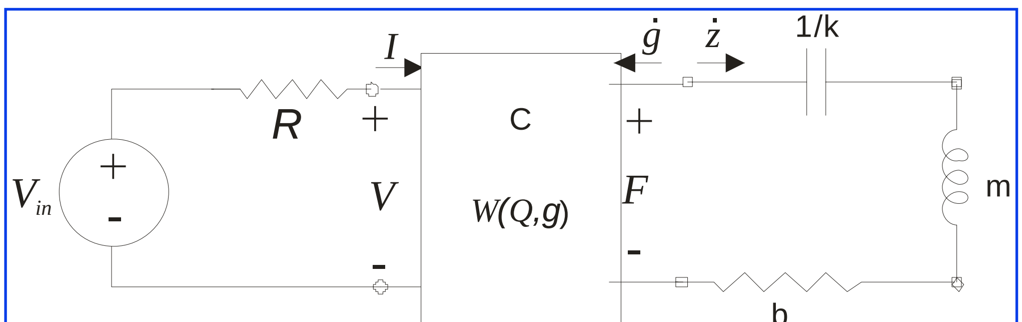
> Addendum: Review of 2^{nd}-order system dynamics

Quasistatic analysis

$$\frac{d}{dt}\begin{bmatrix} Q \\ g \\ \dot{g} \end{bmatrix} = \begin{bmatrix} \frac{1}{R}\left(V_{in} - \frac{Qg}{\varepsilon A}\right) \\ \dot{g} \\ -\frac{1}{m}\left[\frac{Q^2}{2\varepsilon A} - k(g_0 - g) + b\dot{g}\right] \end{bmatrix}$$

State eqns

(Just now)



Equivalent circuit

Fixed-point analysis

Given *static* V_{in}, etc.

What is deflection, charge, etc.?

Follow causal path (Wed)

State equations

State variables

Inputs

$$\dot{\mathbf{x}} = f(\mathbf{x},\mathbf{u})$$
$$\mathbf{y} = g(\mathbf{x},\mathbf{u})$$

Outputs

For transverse electrostatic actuator:

State variables: $\mathbf{x} = \begin{bmatrix} Q \\ g \\ \dot{g} \end{bmatrix}$

Inputs: $\mathbf{u} = [V_{in}]$

$$f(\mathbf{x},\mathbf{u}) = \begin{bmatrix} \frac{1}{R}\left(V_{in} - \frac{Qg}{\varepsilon A}\right) \\ \dot{g} \\ -\frac{1}{m}\left[\frac{Q^2}{2\varepsilon A} - k(g_0 - g) + b\dot{g}\right] \end{bmatrix}$$

Outputs: $\mathbf{y} = [\dot{g}] = g(\mathbf{x},\mathbf{u})$

Fixed points

> **Definition of a fixed point**

- **Solution of *f*(x,u) = 0**
- **time derivatives → 0**

> **Global fixed point**

- **A fixed point when u = 0**
- **Systems can have multiple global fixed points**
- **Some might be stable, others unstable (consider a pendulum)**

> **Operating point**

- **Fixed point when u is a non-zero constant**

Fixed points of the electrostatic actuator

- This analysis is analogous to what we did last time…

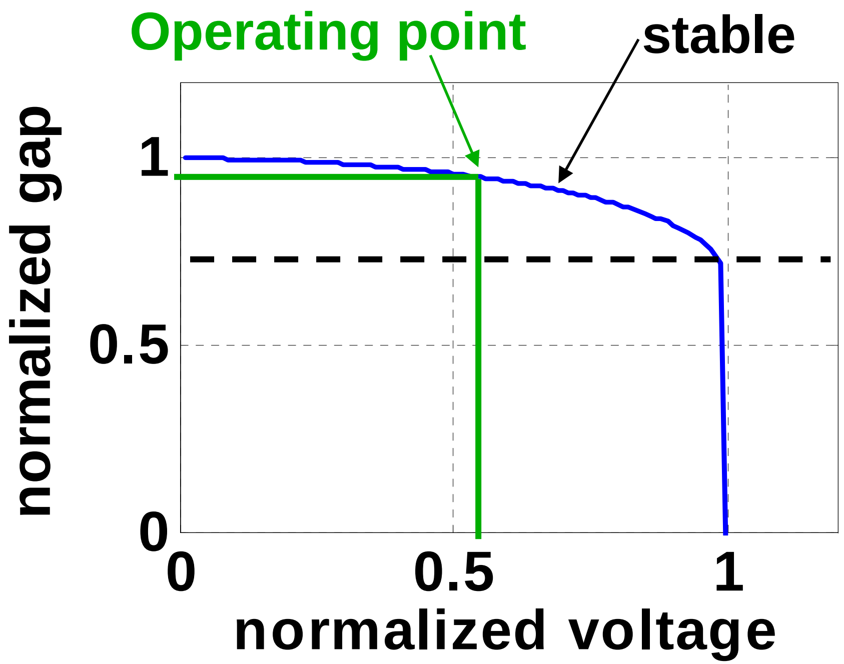
$$0 = \frac{1}{R}\left(V_{in} - \frac{Qg}{\varepsilon A}\right)$$

$$0 = \dot{g}$$

$$0 = -\frac{1}{m}\left[\frac{Q^2}{2\varepsilon A} - k(g_0 - g) + b\dot{g}\right]$$

$$V_{in} = \frac{Qg}{\varepsilon A}$$

$$\frac{Q^2}{2\varepsilon A} = \boxed{\frac{V_{in}^2 \varepsilon A}{2g^2} = k(g_0 - g)}$$



Last time…

$$\boxed{g = g_0 - \frac{\varepsilon A V_{in}^2}{2kg^2}}$$

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Large-signal analysis

$$\frac{d}{dt}\begin{bmatrix} Q \\ g \\ \dot{g} \end{bmatrix} = \begin{bmatrix} \frac{1}{R}\left(V_{in} - \frac{Qg}{\varepsilon A}\right) \\ \dot{g} \\ -\frac{1}{m}\left[\frac{Q^2}{2\varepsilon A} - k(g_0 - g) + b\dot{g}\right] \end{bmatrix}$$

State eqns

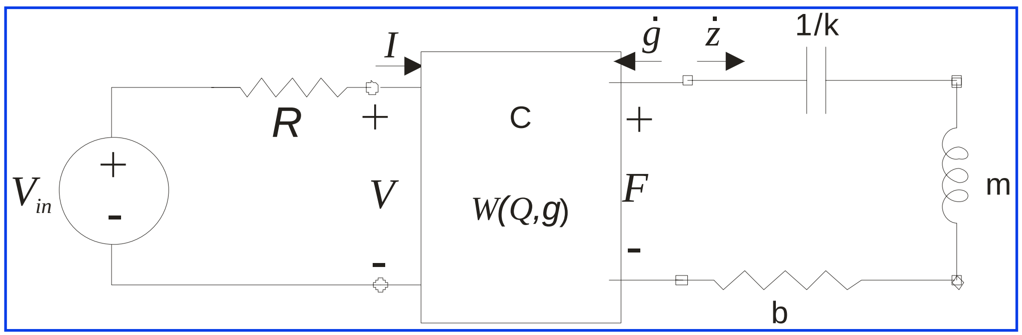
Integrate state eqns

Given a step input $V_{in}(t)u(t)$

What is g(t), Q(t), etc.?

(Earlier)

SPICE



Equivalent circuit

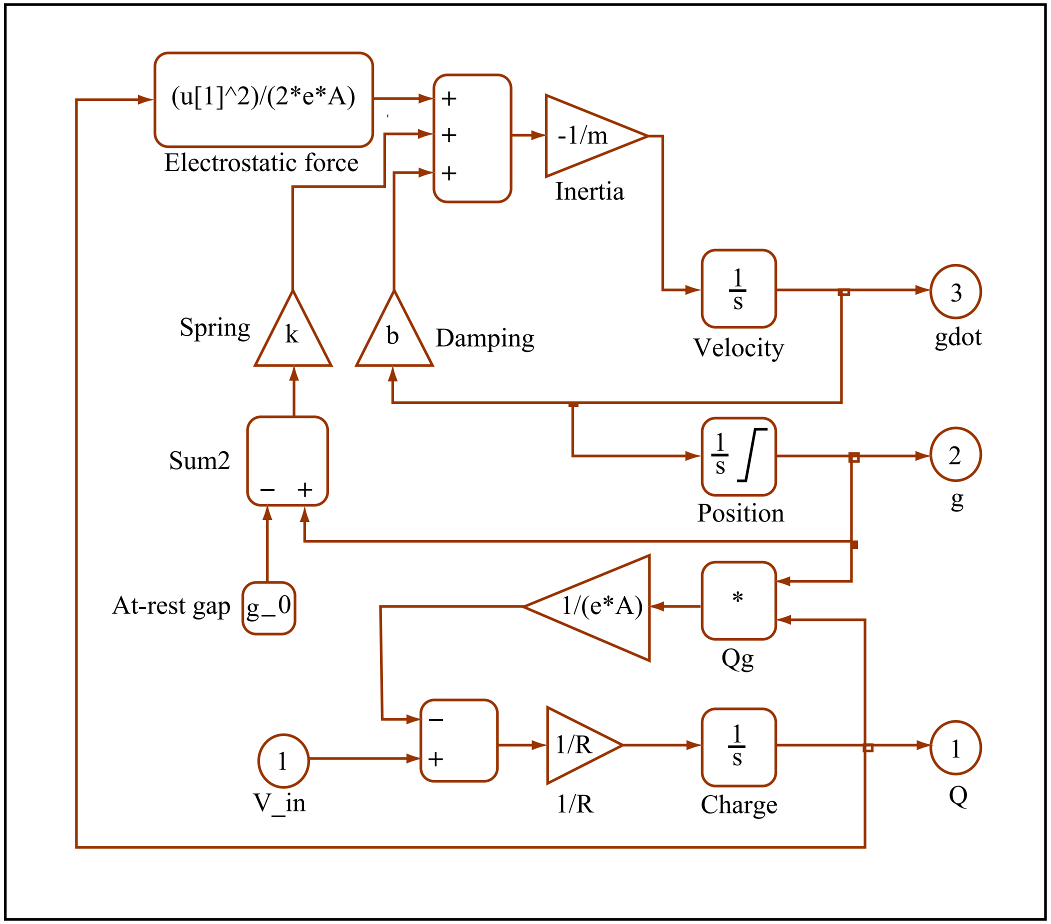
Direct Integration

- This is a brute force approach: integrate the state equations
  - Via MATLAB® (ODExx)
  - Via Simulink®
- We show the SIMULINK® version here
  - Matlab® version later

Electrostatic actuator in Simulink®

$$\mathbf{x} = \begin{bmatrix} Q \\ g \\ \dot{g} \end{bmatrix}, \mathbf{u} = \begin{bmatrix} V_{in} \end{bmatrix}$$

$$\mathbf{f}(\mathbf{x},\mathbf{u}) = \begin{bmatrix} \frac{1}{R}\left(V_{in} - \frac{Qg}{\varepsilon A}\right) \\ \dot{g} \\ -\frac{1}{m}\left[\frac{Q^2}{2\varepsilon A} - k(g_0 - g) + b\dot{g}\right] \end{bmatrix}$$



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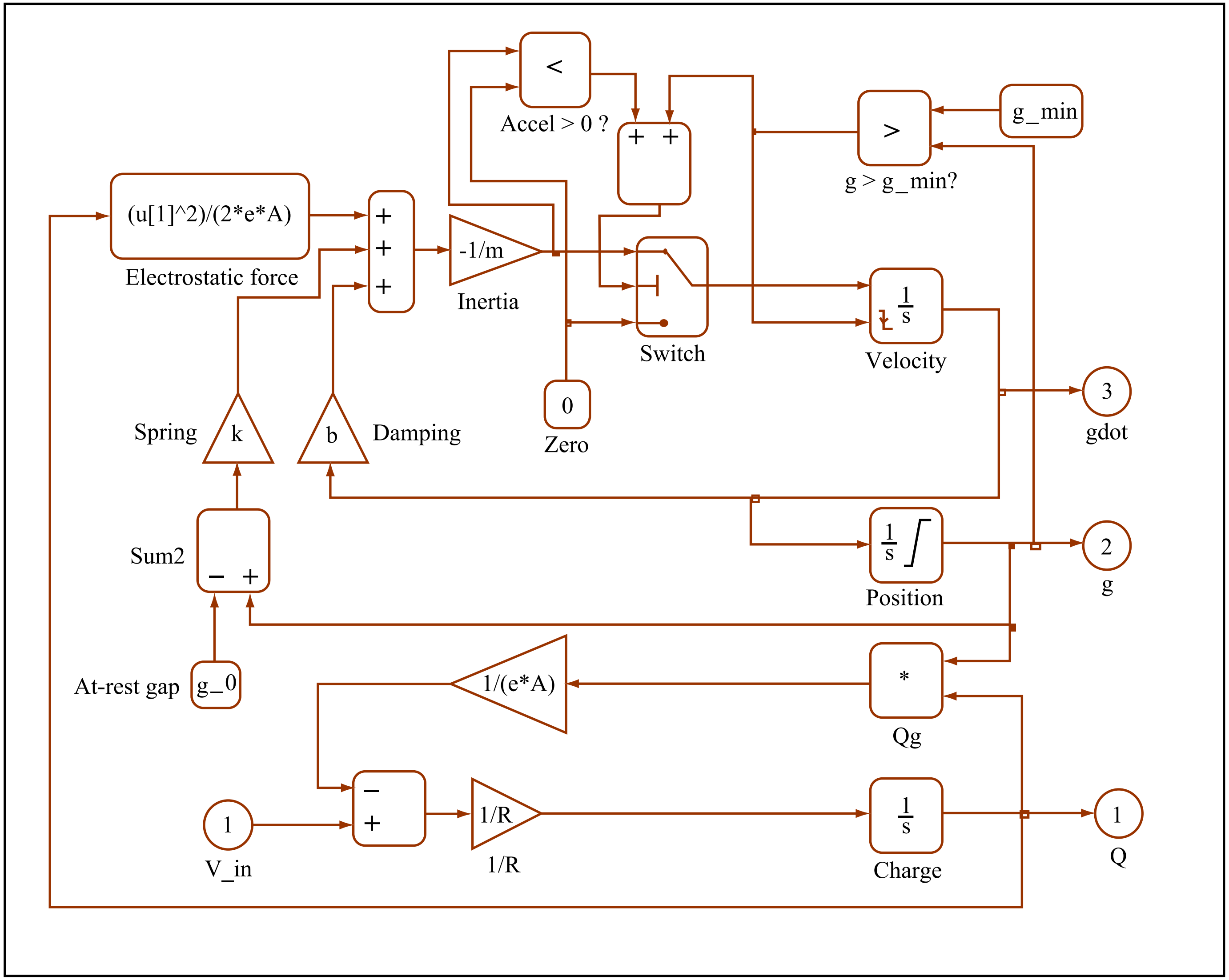
Electrostatic actuator with contact



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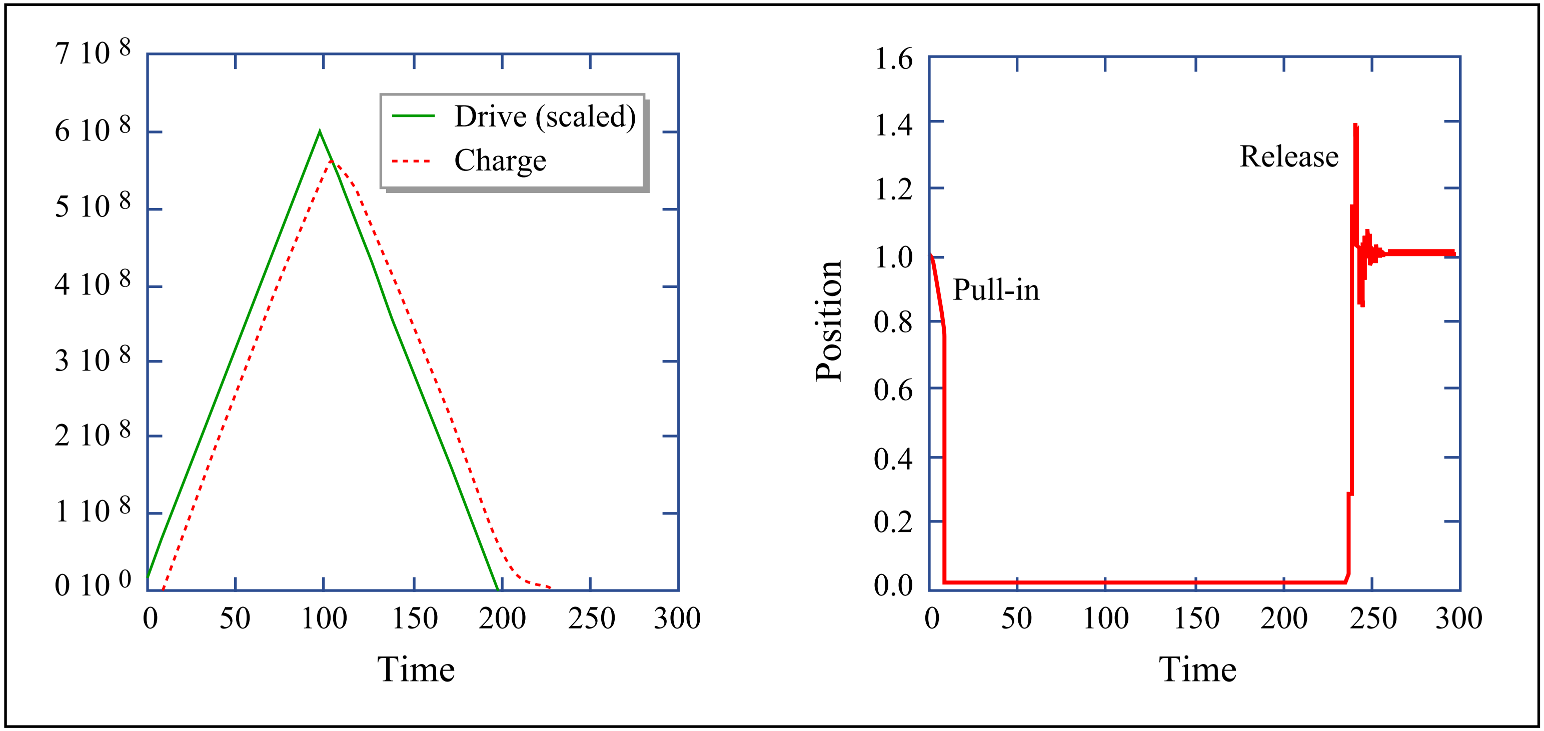
Behavior through pull-in



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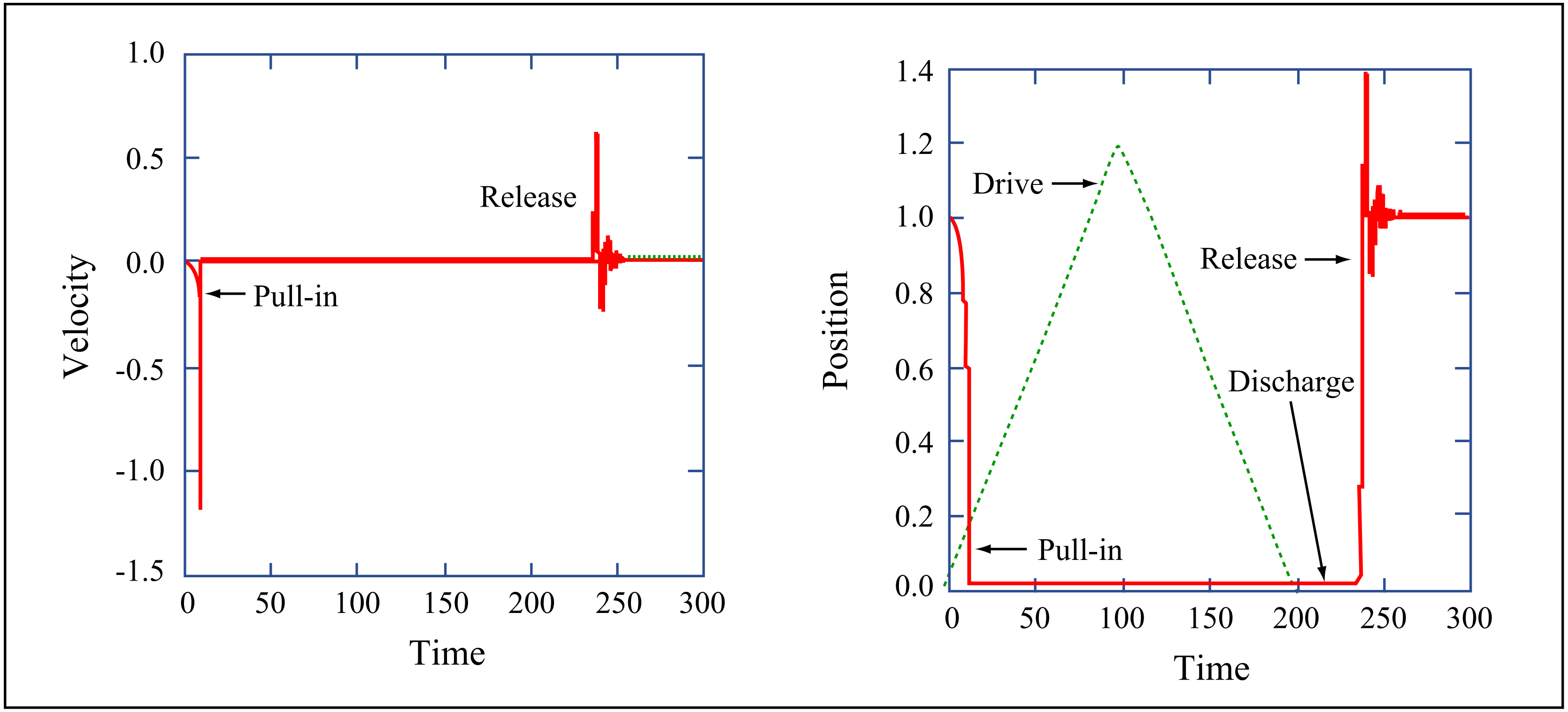
Behavior through pull-in



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> **Small-signal analysis**

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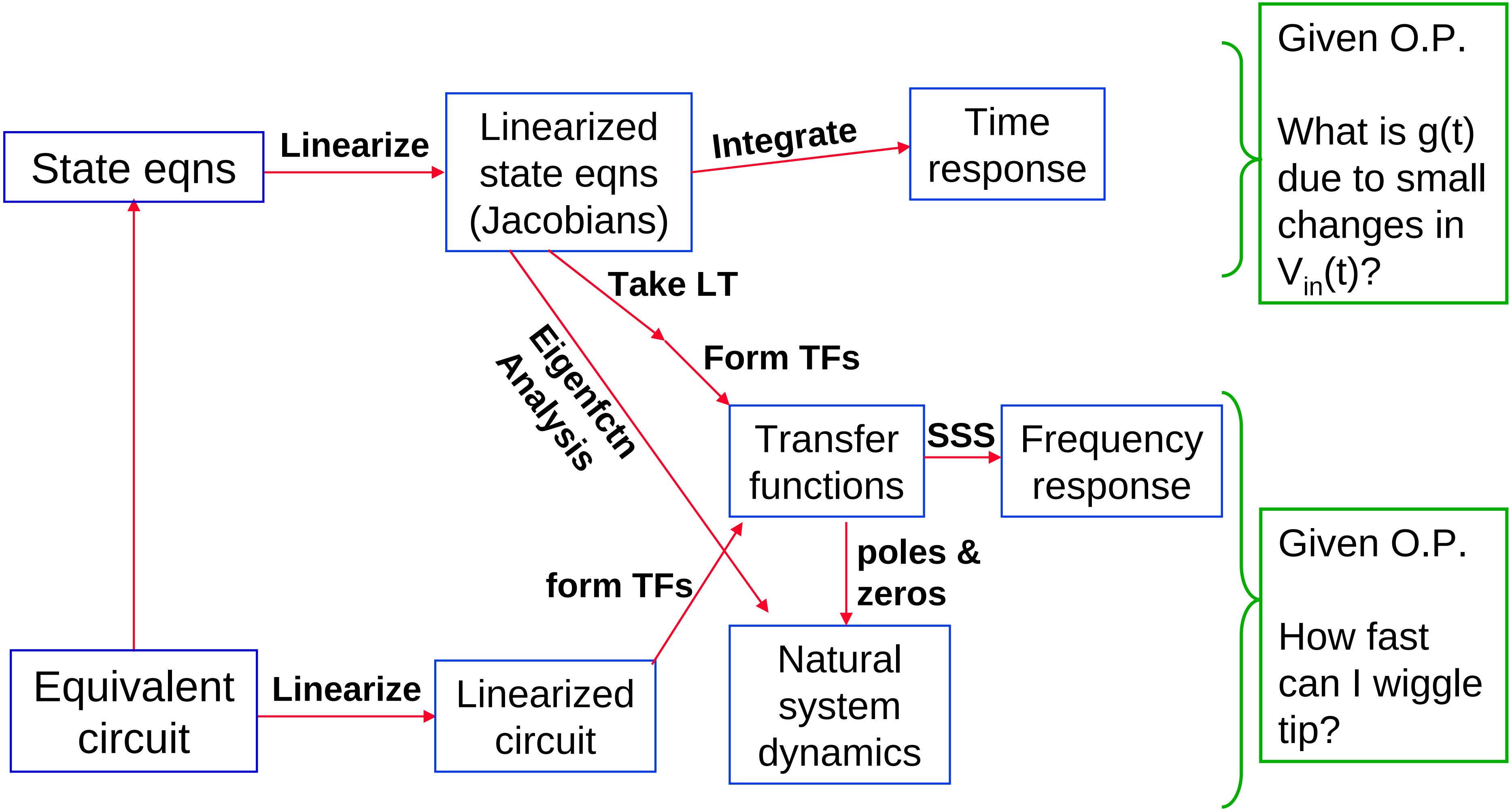
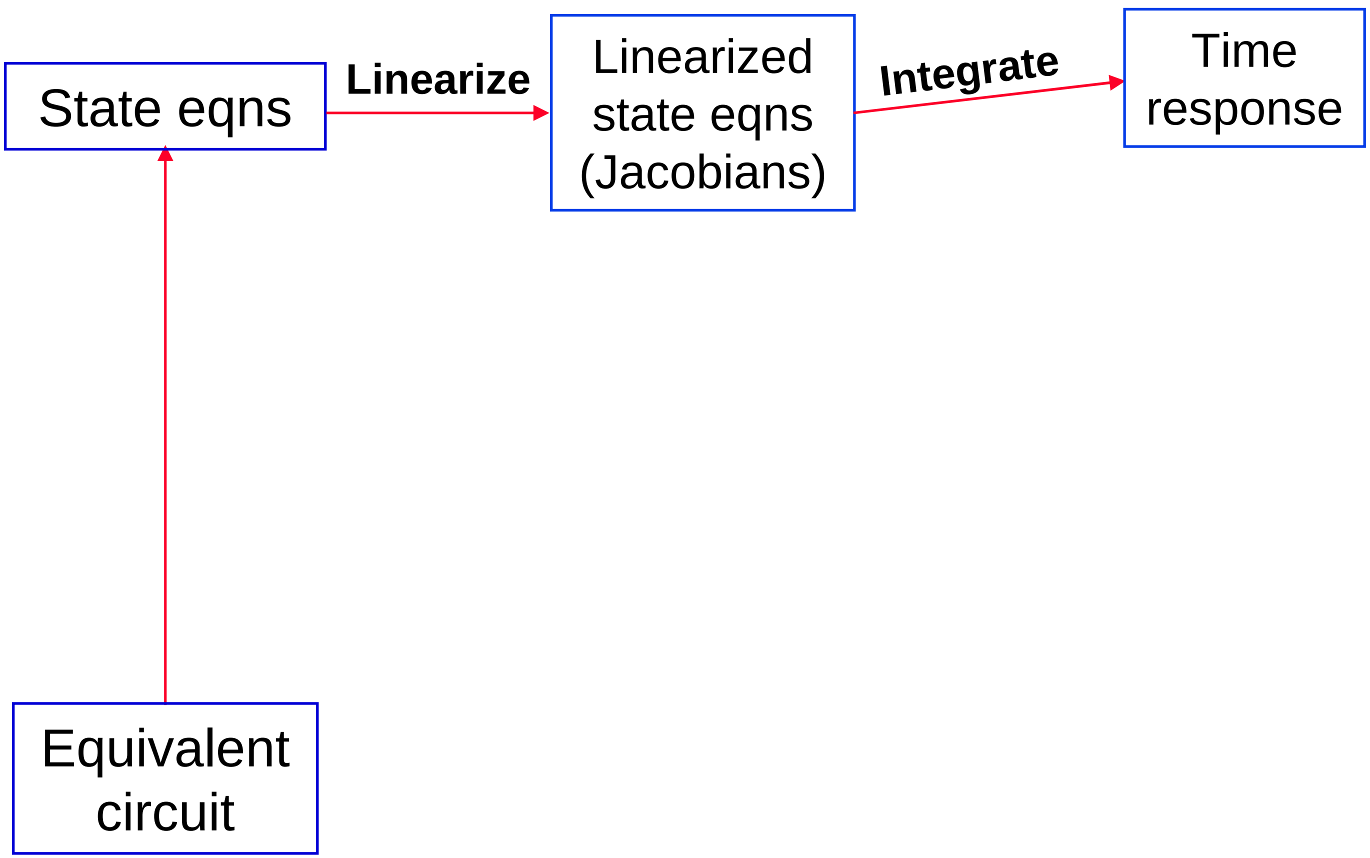
Small-signal analysis



Small-signal analysis



Given O.P.

What is g(t) due to small changes in $V_{in}(t)$?

Given O.P.

How fast can I wiggle tip?

Linearization about a fixed point

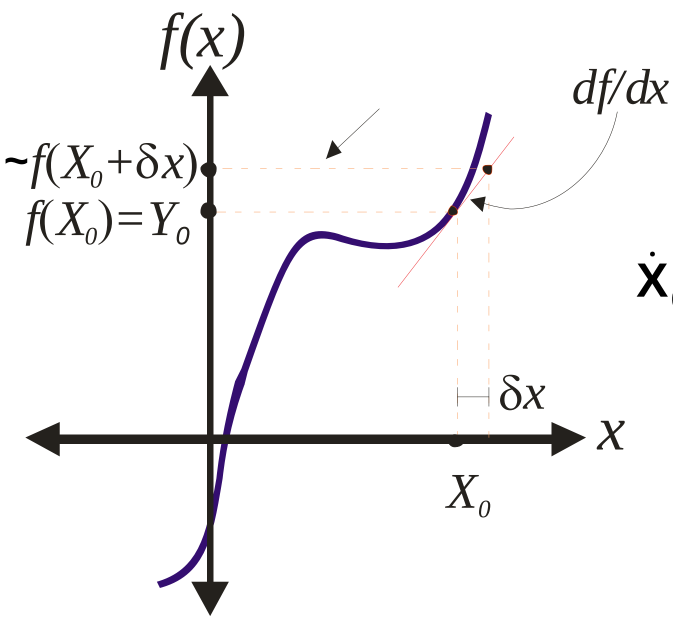
> **This is EXTREMELY common in MEMS literature**

> **This is also done in many other fields, with different names**

- **Small-signal analysis**
- **Incremental analysis**
- **Etc.**

Linearization About an Operating Point

- Using Taylor's theorem, a system can be linearized about any fixed point
- We can do this in one dimension or many



$$f(X_0 + \delta x) \approx f(X_0) + \left.\frac{df}{dx}\right|_{X_0} \delta x$$

$$\dot{\mathbf{x}} = f(\mathbf{x}, \mathbf{u})$$

$$\mathbf{x}(t) = \mathbf{X_0} + \delta\mathbf{x}(t)$$
$$\mathbf{u}(t) = \mathbf{U_0} + \delta\mathbf{u}(t)$$

Operating point

$$\dot{\mathbf{X}}_0 + \frac{d(\delta\mathbf{x})}{dt} = f(\mathbf{X_0} + \delta\mathbf{x}, \mathbf{U_0} + \delta\mathbf{u})$$

Multi-dimensional Taylor

$$\dot{\mathbf{X}}_0 + \frac{d(\delta\mathbf{x})}{dt} = f(\mathbf{X_0}, \mathbf{U_0}) + \left(\left.\frac{\partial f_i}{\partial x_j}\right|_{X_0,U_0}\right)\delta\mathbf{x}(t) + \left(\left.\frac{\partial f_i}{\partial u_j}\right|_{X_0,U_0}\right)\delta\mathbf{u}(t)$$

Cancel

$$\frac{d(\delta x_i(t))}{dt} = \underbrace{\left(\left.\frac{\partial f_i}{\partial x_j}\right|_{X_0,U_0}\right)}_{\mathbf{J_1}}\delta x_i(t) + \underbrace{\left(\left.\frac{\partial f_i}{\partial u_j}\right|_{X_0,U_0}\right)}_{\mathbf{J_2}}\delta u_i(t)$$

Linearization About an Operating Point

- **The resulting set of equations are linear, and have dynamics described by the Jacobians of *f*(x,u) evaluated at the fixed point.**
- **These describe how much a small change in one state variable affects itself or another state variable**
- **The O.P. must be evaluated to use the Jacobian**
- **Example – linearization of the voltage-controlled electrostatic actuator**

$$\delta\dot{\mathbf{x}}(\mathbf{t}) = \mathbf{J_1}\delta\mathbf{x} + \mathbf{J_2}\delta\mathbf{u}(\mathbf{t})$$

$$\mathbf{J}_1 = \begin{bmatrix} \left.\frac{\partial f_1}{\partial Q}\right|_{O.P.} & \left.\frac{\partial f_1}{\partial g}\right|_{O.P.} & \left.\frac{\partial f_1}{\partial \dot{g}}\right|_{O.P.} \\ \left.\frac{\partial f_2}{\partial Q}\right|_{O.P.} & \left.\frac{\partial f_2}{\partial g}\right|_{O.P.} & \left.\frac{\partial f_2}{\partial \dot{g}}\right|_{O.P.} \\ \left.\frac{\partial f_3}{\partial Q}\right|_{O.P.} & \left.\frac{\partial f_3}{\partial g}\right|_{O.P.} & \left.\frac{\partial f_3}{\partial \dot{g}}\right|_{O.P.} \end{bmatrix}$$

$$\frac{d}{dt}\begin{pmatrix} \delta Q \\ \delta g \\ \delta \dot{g} \end{pmatrix} = \underbrace{\begin{pmatrix} -\frac{\hat{g}_0}{R\varepsilon A} & -\frac{Q_0}{R\varepsilon A} & 0 \\ 0 & 0 & 1 \\ -\frac{Q_0}{m\varepsilon A} & -\frac{k}{m} & -\frac{b}{m} \end{pmatrix}}_{\mathbf{J_1}} \begin{pmatrix} \delta Q \\ \delta g \\ \delta \dot{g} \end{pmatrix} + \underbrace{\begin{pmatrix} \frac{1}{R} \\ 0 \\ 0 \end{pmatrix}}_{\mathbf{J_2}} (\delta V_{in})$$

State Equations for Linear Systems

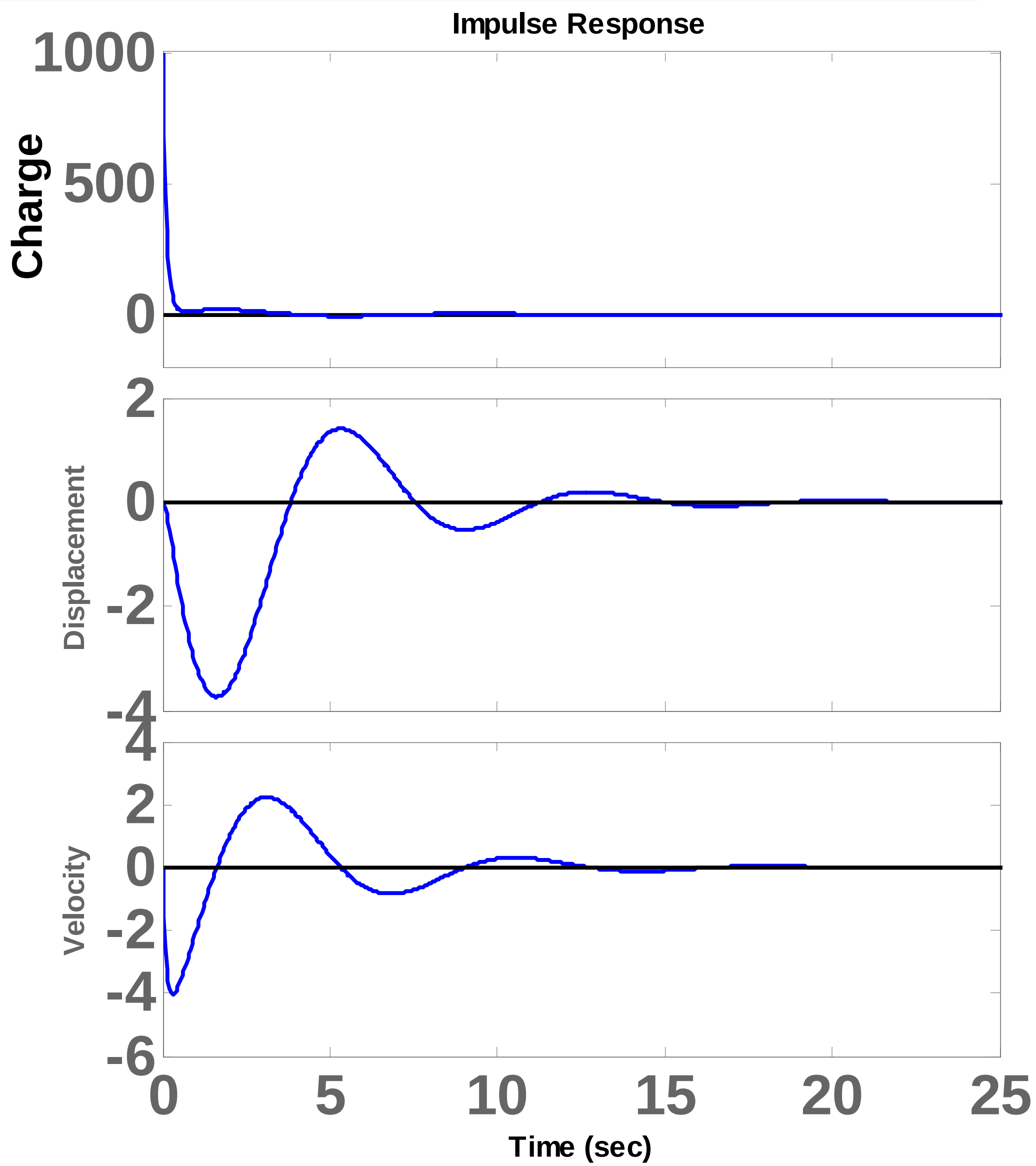
- Normally expressed with:
  - x: a vector of state variables
  - u: a vector of inputs
  - y: a vector of outputs
  - Four matrices, A,B,C, D

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$
$$\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}$$

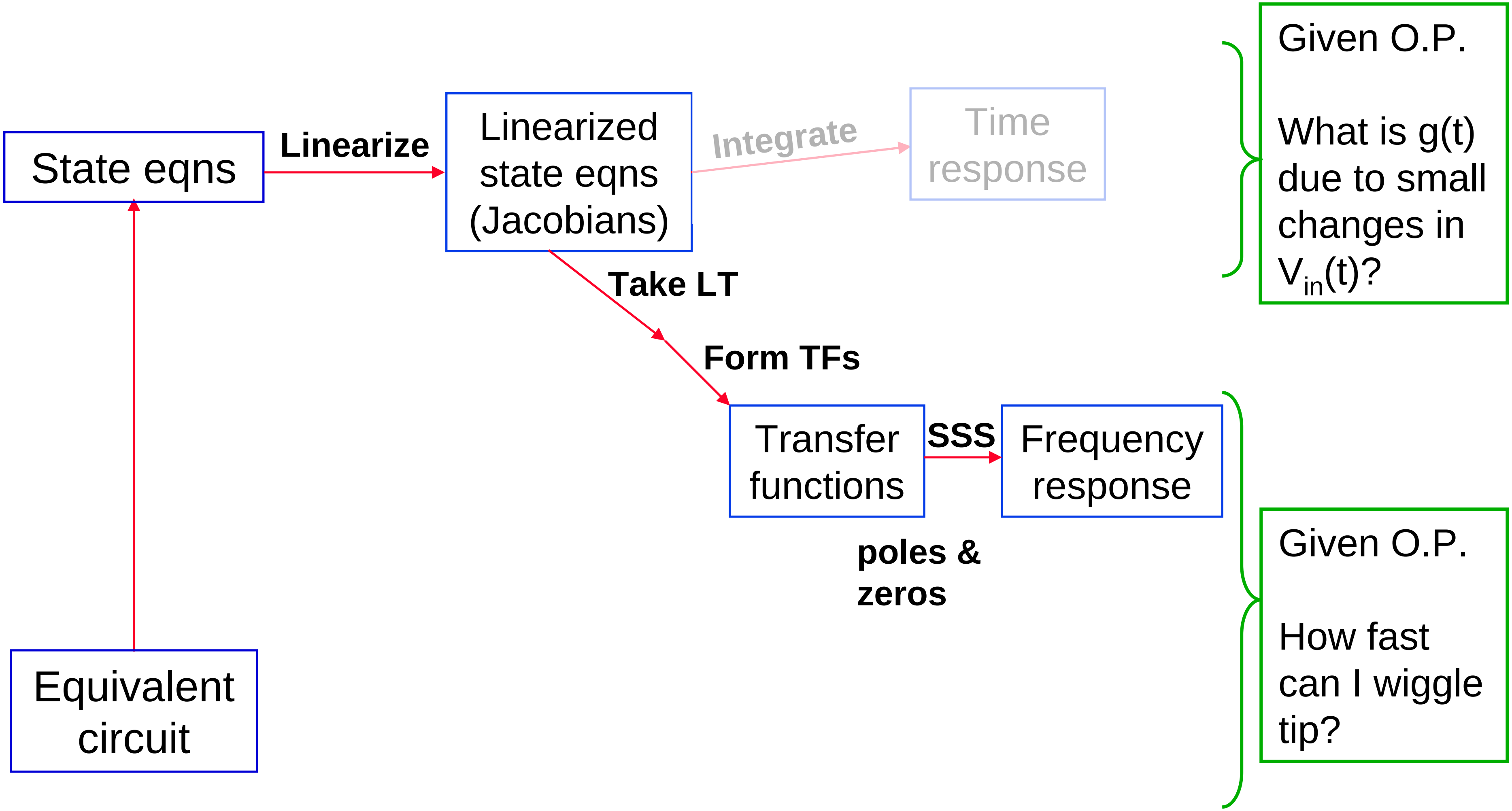
- For us, Jacobian matrices take the place of A and B
- C and D depend on what outputs are desired
  - Often C is identity and D is zero
- Can use to simulate time responses to arbitrary ***SMALL*** inputs
  - Remember, this is only valid for small deviations from O.P.

Direct Integration in Time

- Can integrate via Simulink® model (as before) or MATLAB®
- First define system in MATLAB®
  - using `ss(J1,J2,C,D)` or alternate method
- Can use MATLAB® commands `step`, `initial`, `impulse` etc.
- Response of electrostatic actuator to impulse of voltage
  - Parameters from text (pg 167)



Small-signal analysis



Solve via Laplace transform

- **Use Laplace Transforms to solve in frequency domain**
  - **Transform DE to algebraic equations**
  - **Use unilateral Laplace to allow for non-zero IC's**

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$

$$\Downarrow \textit{Unilateral Laplace}$$

$$s\mathbf{X}(s) - \mathbf{x}(0) = \mathbf{A}\mathbf{X}(s) + \mathbf{B}\mathbf{U}(s)$$

$$s\mathbf{I}\mathbf{X}(s) - \mathbf{x}(0) = \mathbf{A}\mathbf{X}(s) + \mathbf{B}\mathbf{U}(s)$$

$$\left(s\mathbf{I} - \mathbf{A}\right)\mathbf{X}(s) = \mathbf{x}(0) + \mathbf{B}\mathbf{U}(s)$$

Transfer Functions

- Transfer functions H(s) are useful for obtaining compact expression of input-output relation
  - What is the tip displacement as a function of voltage
- Most easily obtained from equivalent circuit
- But can also be obtained from linearized state eqns
  - Depends on A, B, C (or J_1, J_2, C) matrices
  - Can do this for fun analytically (see attachment at end)
  - Matlab can automatically convert from s.s to t.f. formulations
- For our actuator, we would get three transfer functions

$$\mathbf{H}(s) = \begin{bmatrix} \dfrac{Q(s)}{\mathbf{V}_{in}(s)} \\ \dfrac{\mathbf{g}(s)}{\mathbf{V}_{in}(s)} \\ \dfrac{\dot{\mathbf{g}}(s)}{\mathbf{V}_{in}(s)} \end{bmatrix}$$

Sinusoidal Steady State

- **When a LTI system is driven with a sinusoid, the steady-state response is a sinusoid at the same frequency**
- **The amplitude of the response is |H(jω)|**
- **The phase of the response relative to the drive is the angle of H(jω)**
- **A plot of log magnitude vs log frequency and angle vs log frequency is called a Bode plot**

$$u(t) = U_0 \cos(\omega t)$$

$$Y(j\omega) = H(j\omega) U(j\omega)$$

$$y_{sss}(t) = Y_0 \cos(\omega t + \theta)$$

$$Y_0 = \left|H(j\omega)\right| U_0$$

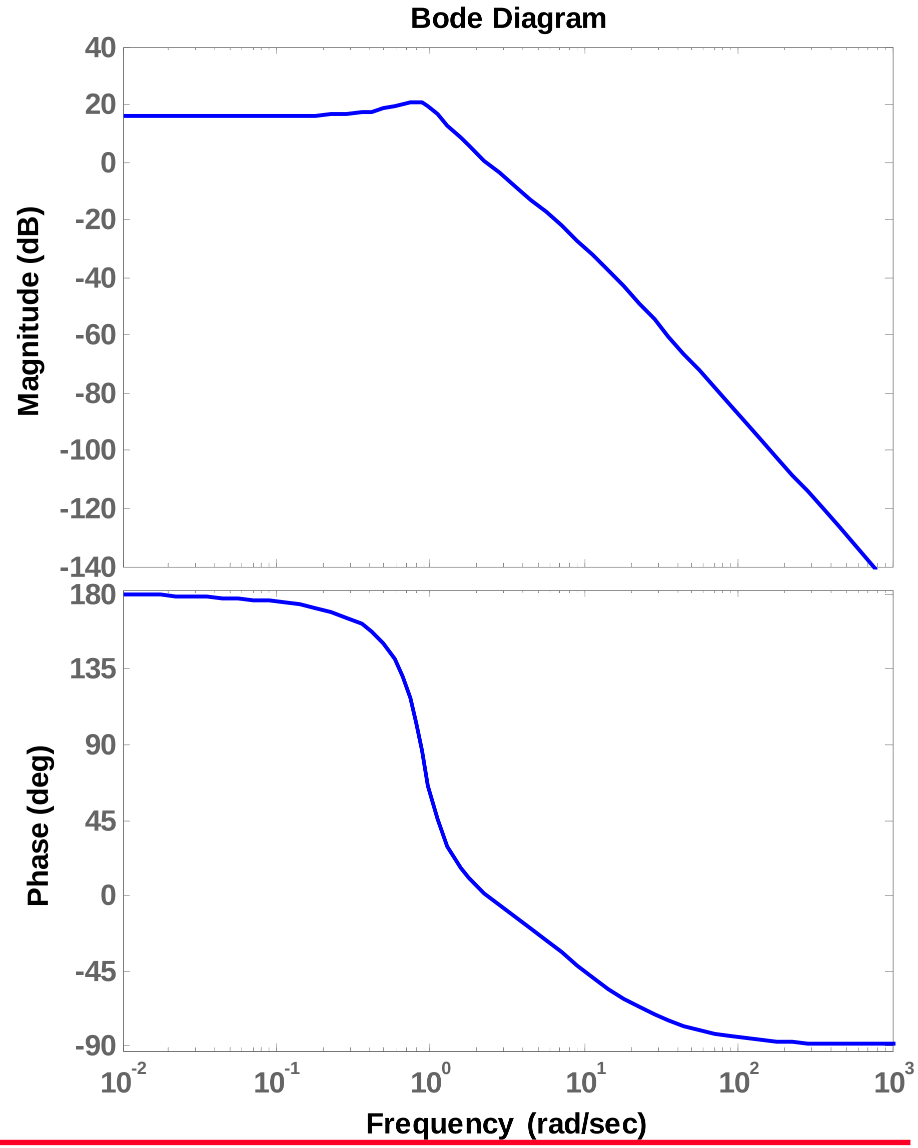
$$\tan\theta = \frac{\mathrm{Im}\left\{H(j\omega)\right\}}{\mathrm{Re}\left\{H(j\omega)\right\}}$$

Bode plot of electrostatic actuator

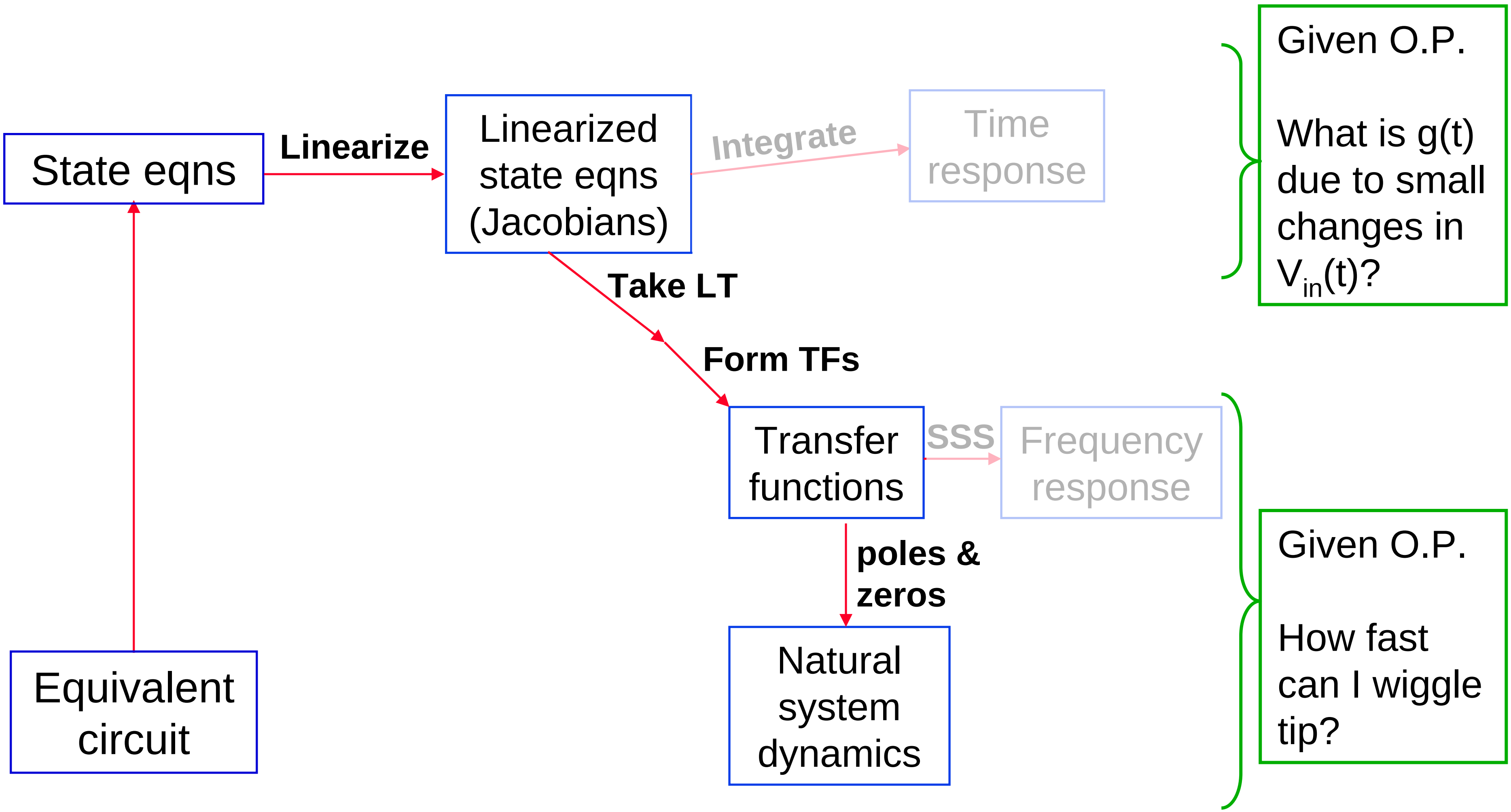
- Use Matlab® command **bode** with previously defined system **sys**
- Evaluate only one of TFs

$$\mathbf{H}(s) = \frac{\mathbf{g}(s)}{\mathbf{V}_{\mathbf{in}}(s)}$$

- This tells us how quickly we can wiggle tip!
  - At a certain OP!



Small-signal analysis



Poles and Zeros

- For our models, system function is a ratio of polynomials in *s*
- Roots of denominator are called poles
  - They describe the natural (unforced) response of the system
- Roots of the numerator are called zeros
  - They describe particular frequencies that fail to excite any output
- System functions with the same poles and zeros have the same dynamics

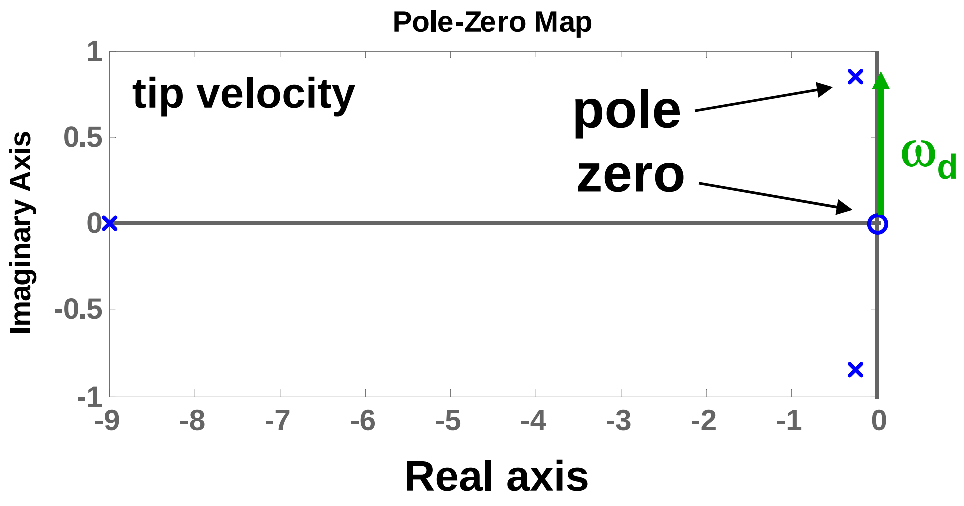
$$\mathbf{H}(s) = \frac{\mathbf{g}(s)}{\mathbf{V}_{\mathbf{in}}(s)} = \frac{\dfrac{-Q_0}{\varepsilon ARm}}{s^3 + \left(\dfrac{1}{RC_0} + \dfrac{b}{m}\right)s^2 + \left(\dfrac{1}{RC_0}\dfrac{b}{m} + \dfrac{k}{m}\right)s + \left(\dfrac{1}{RC_0}\dfrac{k}{m} - \dfrac{Q_0^2}{\varepsilon^2 A^2}\dfrac{1}{Rm}\right)}$$

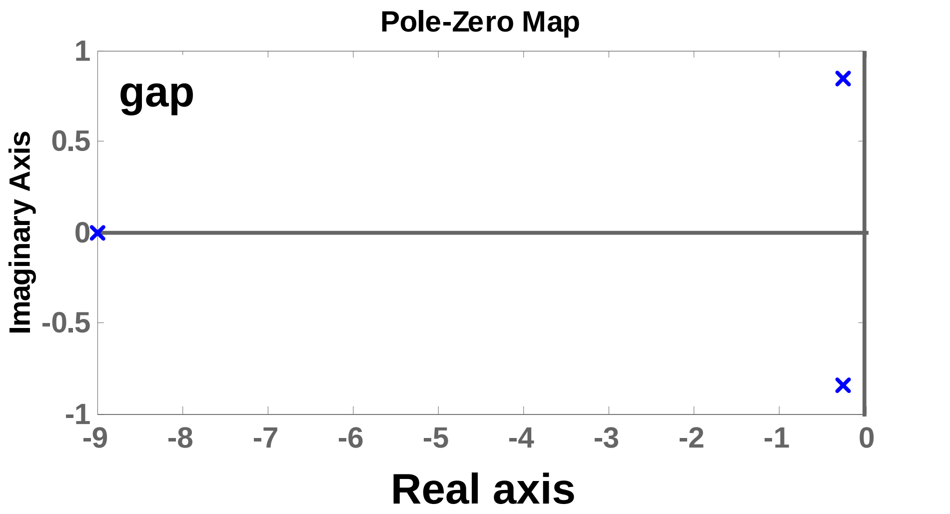
where $C_0 = \dfrac{\varepsilon A}{\hat{g}_0}$

- MATLAB solution for poles is VERY long

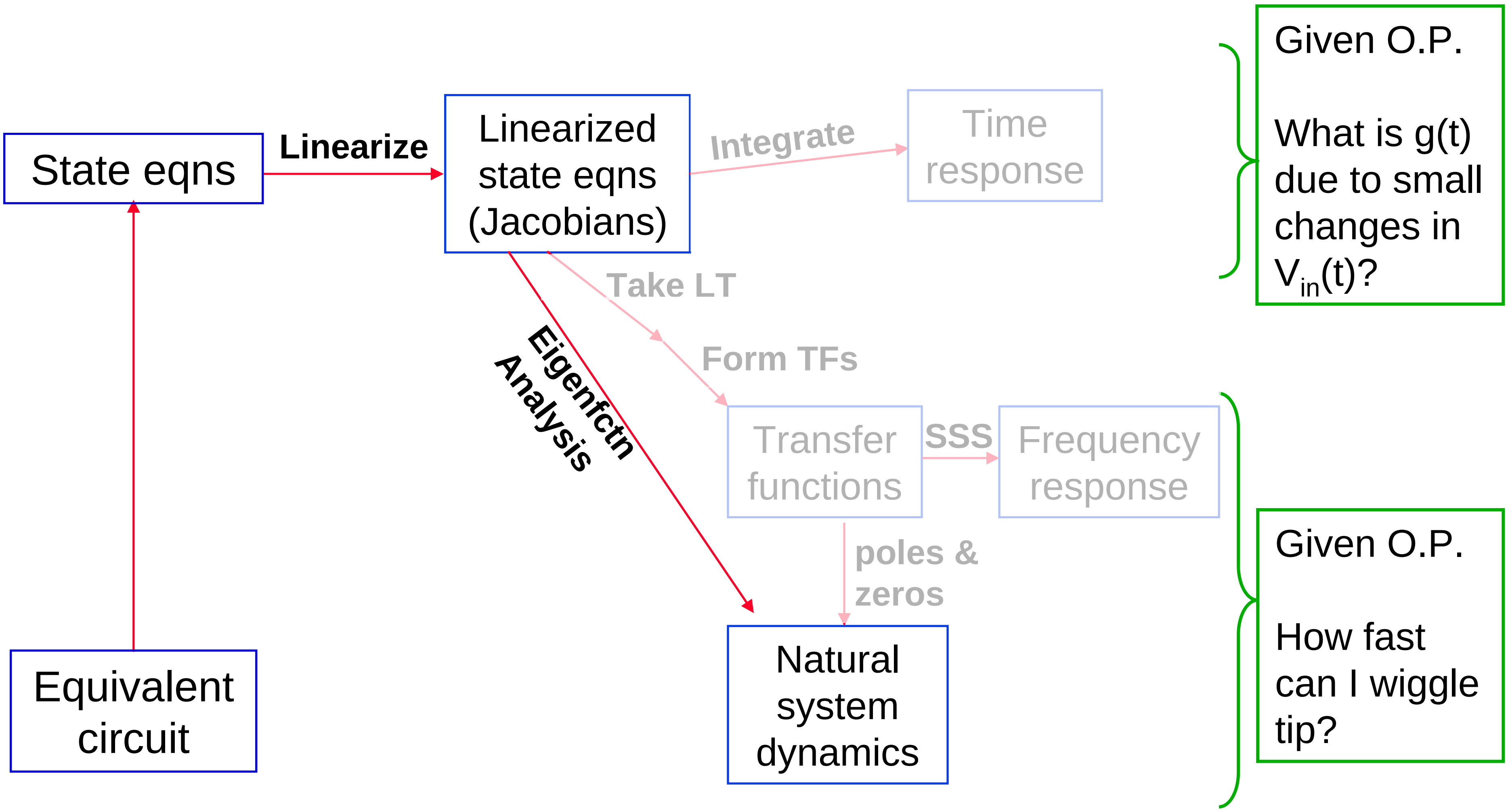
Pole-zero diagram

- Displays information about dynamics of system function
  - Matlab command `pzmap`
- Useful for examining dynamics, stability, etc.





Small-signal analysis



Eigenfunction Analysis

- For an LTI system, we can find the eigenvalues and eigenvectors of the J_1 (or A) matrix describing the internal dynamics

For scalar 1st-order system:

$$\frac{dx}{dt} = \lambda x \quad \Rightarrow x(t) = K_0 e^{\lambda t} + K_1$$

Our linear (or linearized) homogeneous systems look like:

$$\delta\dot{\mathbf{x}}(\mathbf{t}) = \mathbf{J_1}\delta\mathbf{x} + \mathbf{J_2}\delta\mathbf{u(t)}$$

$$\frac{d\mathbf{x}}{dt} = \mathbf{Ax} \qquad \frac{d(\delta\mathbf{x})}{dt} = \mathbf{J_1}\delta\mathbf{x}$$

If we try solution:

$$\mathbf{x}(t) = \mathbf{K}e^{\lambda t}$$

Plug into DE:

$$\lambda\mathbf{x} = \mathbf{Ax}$$

- This is an eigenvalue equation
- If we find λ we can find natural frequencies of system

Eigenfunction Analysis

> These λ are the same as the poles s_i of the system

> Can solve analytically

- Find λ from $\det(A-\lambda I)=0$

> Or numerically `eig(sys)`

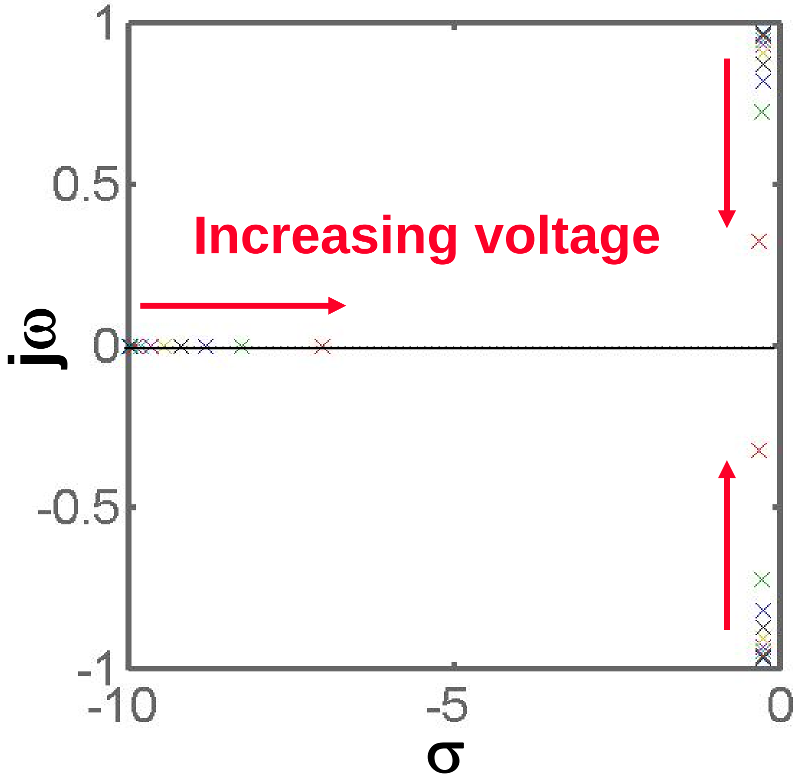
-8.9904

-0.2627 + 0.8455i

-0.2627 - 0.8455i

Linearized system poles

- We can use either λ_i or s_i to determine natural frequencies of system
- As we increase applied voltage
  - Stable damped resonant frequency decreases
- Plotting poles as system changes is a root-locus plot



Spring softening

- Plot damped resonant frequency versus applied voltage
- Resonant frequency is changing because net spring constant *k* changes with frequency
- This is an electrically tuned mechanical resonator

$$k' = k - \frac{\varepsilon A V^2}{g^3}$$

This is called *spring softening*

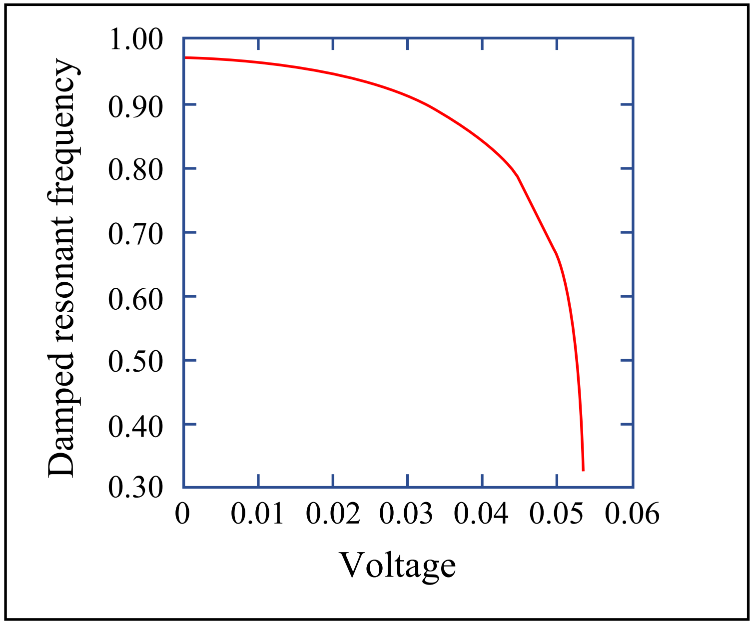
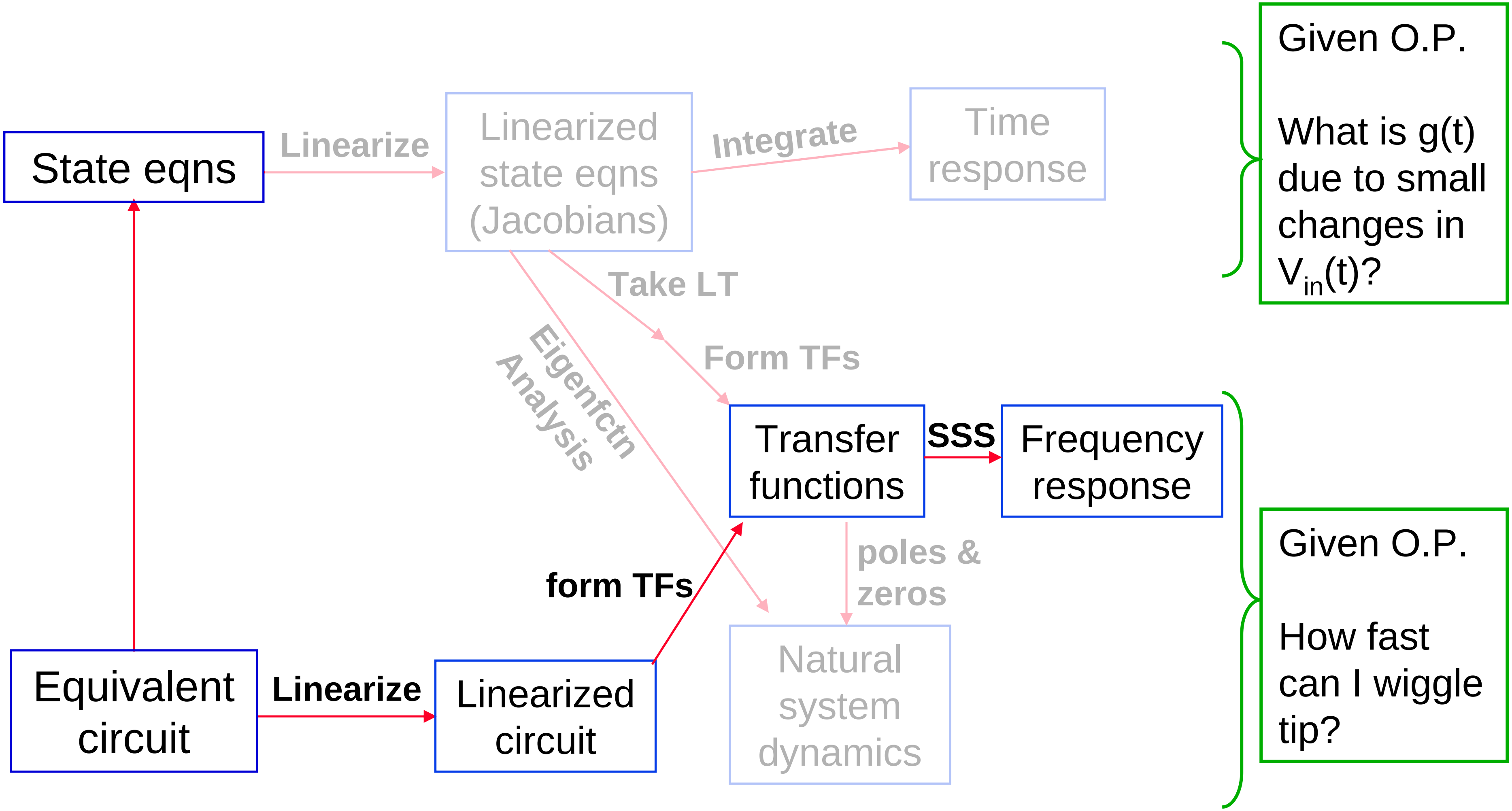


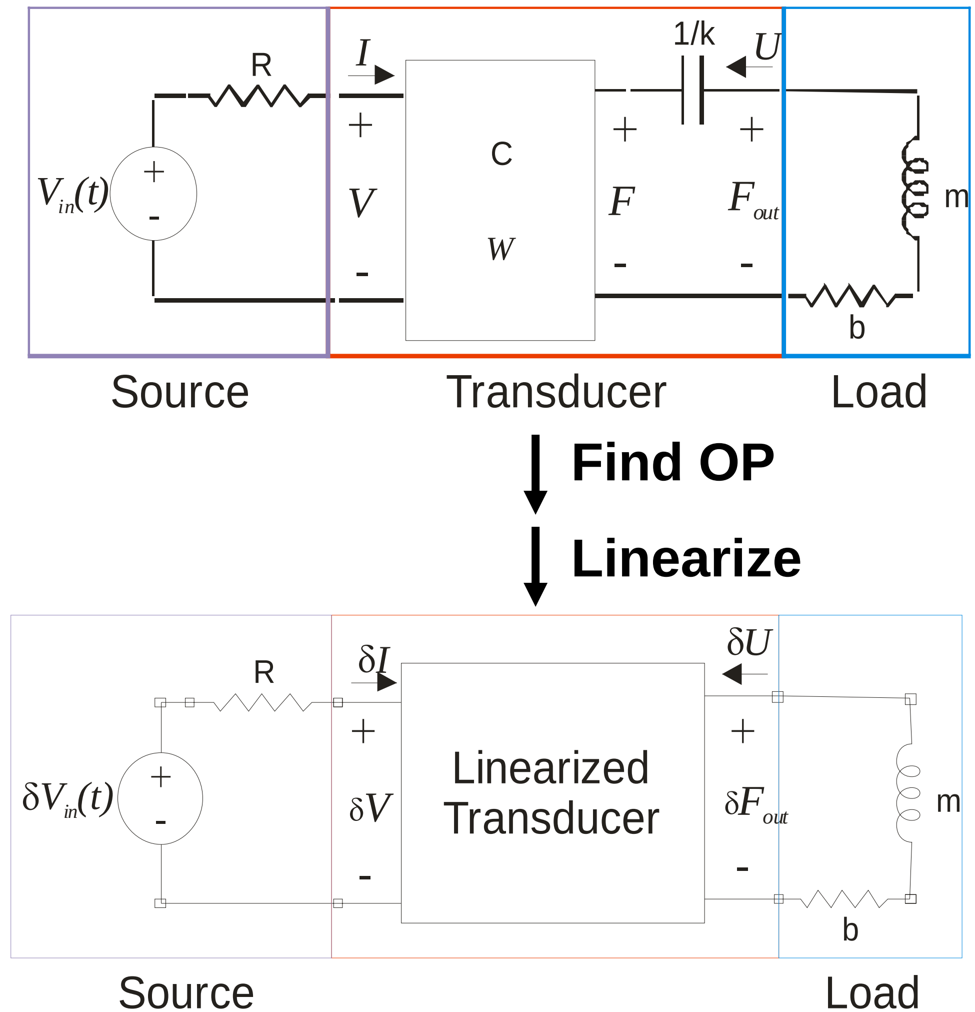
Image by MIT OpenCourseWare.
Adapted from Figure 7.5 in Senturia, Stephen D. *Microsystem Design*. Boston, MA: Kluwer Academic Publishers, 2001, p. 169. ISBN: 9780792372462.

Small-signal analysis



Linearized Transducers

- Can we directly linearize our equivalent circuit? YES!
- This is perhaps the most common analysis in the literature
- First, choose what is load and what is transducer
  - Here we include spring with transducer



Linearized Transducer Model

- **First, find O.P.**

$$V_0,\ \hat{g}_0,\ Q_0$$

- **Next, generate matrix to relate *incremental* port variables to each other**
  - **Start from energy and force relations**

$$V = \frac{Qg}{\varepsilon A}$$

$$F_{out} = \frac{Q^2}{2\varepsilon A} - k(g_0 - g)$$

  - **Linearize (take partials…)**

$$\begin{bmatrix} \delta V \\ \delta F_{out} \end{bmatrix} = \begin{bmatrix} \frac{\hat{g}_0}{\varepsilon A} & \frac{Q_0}{\varepsilon A} \\ \frac{Q_0}{\varepsilon A} & k \end{bmatrix} \begin{bmatrix} \delta Q \\ \delta g \end{bmatrix}$$

- **Recast in terms of port variables**

$$\begin{bmatrix} \delta Q \\ \delta g \end{bmatrix} = \begin{bmatrix} \delta I / s \\ \delta U / s \end{bmatrix}$$

- **Define intermediate variables**

$$C_0 = \frac{\varepsilon A}{\hat{g}_0},\ V_0 = \frac{Q_0}{C_0}$$

- **Final expression**

$$\begin{bmatrix} \delta V \\ \delta F_{out} \end{bmatrix} = \begin{bmatrix} \frac{1}{sC_0} & \frac{V_0}{s\hat{g}_0} \\ \frac{V_0}{s\hat{g}_0} & \frac{k}{s} \end{bmatrix} \begin{bmatrix} \delta I \\ \delta U \end{bmatrix}$$

Linearized Transducers

- Now we want to convert this relation into a circuit
- Many circuit topologies are consistent with this matrix relation
- THIS IS NOT UNIQUE!

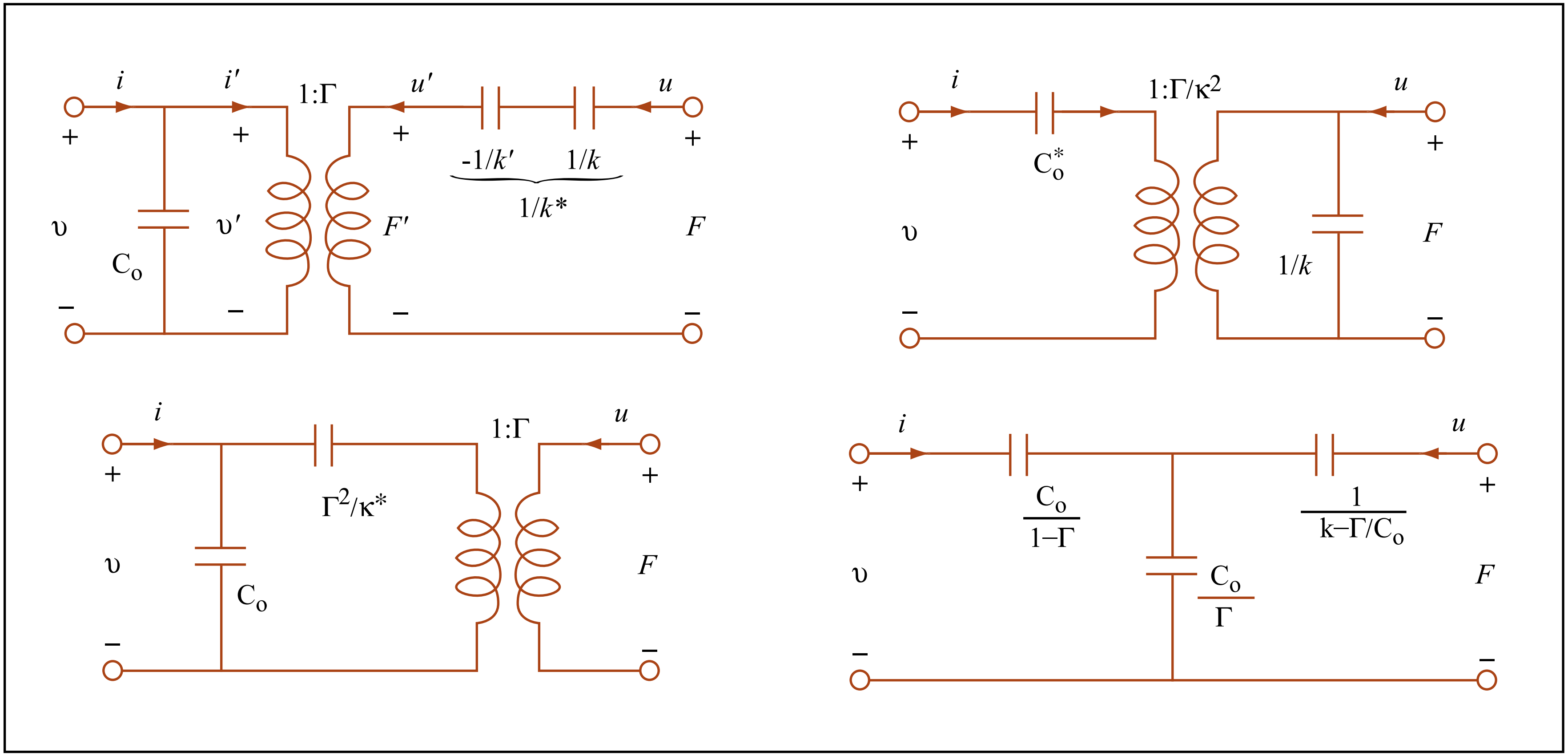


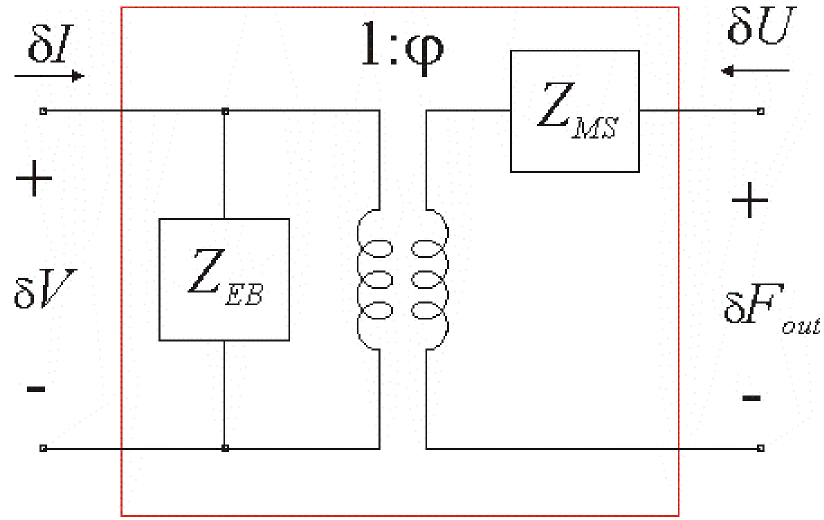
Image by MIT OpenCourseWare.
Adapted from Figure 5 on p. 163 in Tilmans, Harrie A. C. "Equivalent Circuit Representations of Electromechanical Transducers: I. Lumped-parameter Systems." *Journal of Micromechanics and Microengineering* 6, no. 1 (1996): 157-176.

Linearized Transducers

> **This is the one used in the text**

$$\begin{bmatrix} \delta V \\ \delta F_{out} \end{bmatrix} = \begin{bmatrix} Z_{EB} & \varphi Z_{EB} \\ \varphi Z_{EB} & Z_{MO} \end{bmatrix} \begin{bmatrix} \delta I \\ \delta U \end{bmatrix}$$

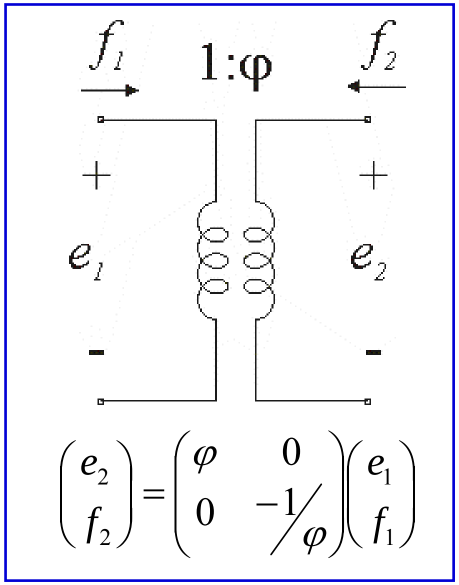
$$Z_{MS} = Z_{MO}\left(1 - \frac{\varphi^2 Z_{EB}}{Z_{MO}}\right)$$



> **Uses a transformer**

- **Transforms port variables**
- **Doesn't store energy**

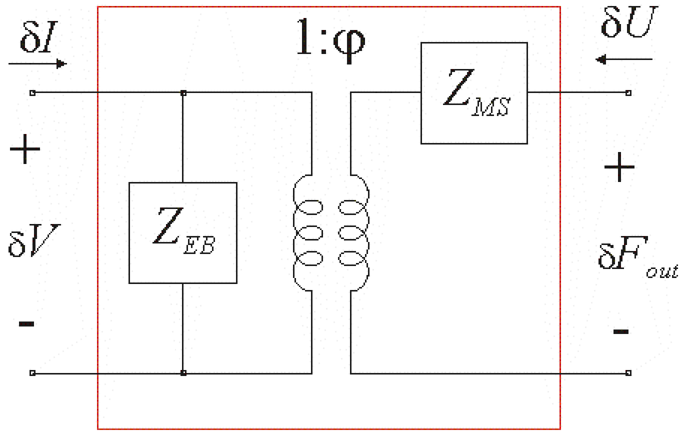
> **What we want to do now is identify Z_{EB}, Z_{MS} and φ, and figure out what they mean…**



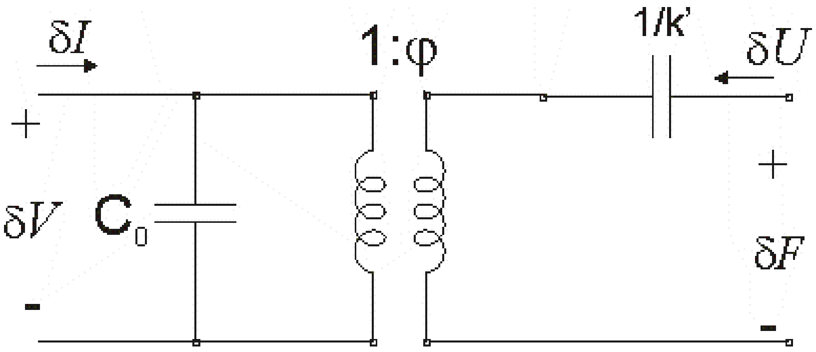
Linearized Transducers

$$\begin{bmatrix} \delta V \\ \delta F_{out} \end{bmatrix} = \begin{bmatrix} Z_{EB} & \varphi Z_{EB} \\ \varphi Z_{EB} & Z_{MO} \end{bmatrix} \begin{bmatrix} \delta I \\ \delta U \end{bmatrix}$$

$$Z_{MS} = Z_{MO}\left(1 - \frac{\varphi^2 Z_{EB}}{Z_{MO}}\right)$$



$$\begin{bmatrix} \delta V \\ \delta F \end{bmatrix} = \begin{bmatrix} \frac{1}{sC_0} & \frac{V_0}{s\hat{g}_0} \\ \frac{V_0}{s\hat{g}_0} & \frac{k}{s} \end{bmatrix} \begin{bmatrix} \delta I \\ \delta U \end{bmatrix}$$



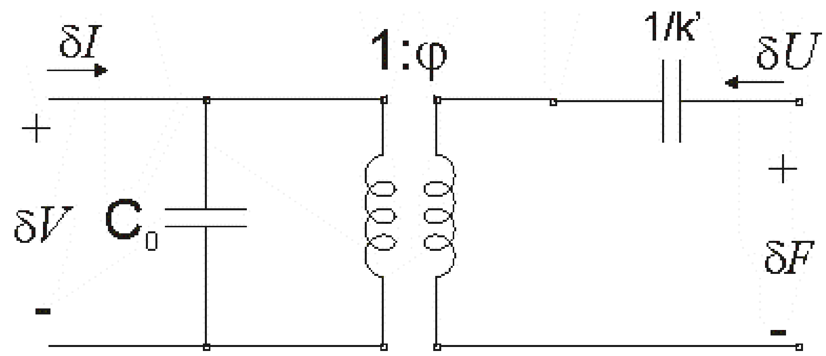
$$Z_{MS} = \frac{k}{s}\left(1 - \left(\frac{Q_0}{\hat{g}_0}\right)^2 \frac{1/sC_0}{k/s}\right) = \frac{k}{s}\left(1 - \left(\frac{Q_0}{\hat{g}_0}\right)^2 \frac{1}{C_0 k}\right)$$

$$= \frac{k}{s}\left(1 - \frac{Q_0^2}{\varepsilon A k \hat{g}_0}\right) \Rightarrow k' = k - \frac{Q_0^2}{\varepsilon A \hat{g}_0}$$

$$\varphi = \frac{C_0 V_0}{\hat{g}_0}$$

Linearized Transducers

- C_0 represents the capacitance of the structure seen from the electrical port
- It is simply the capacitance at the gap given by the operating point
- As V_{in} increases, C_0 will increase until the structure pulls in
- This is a tunable capacitor



$$C_0 = \frac{\varepsilon A}{\hat{g}_0}$$

Linearized Transducers

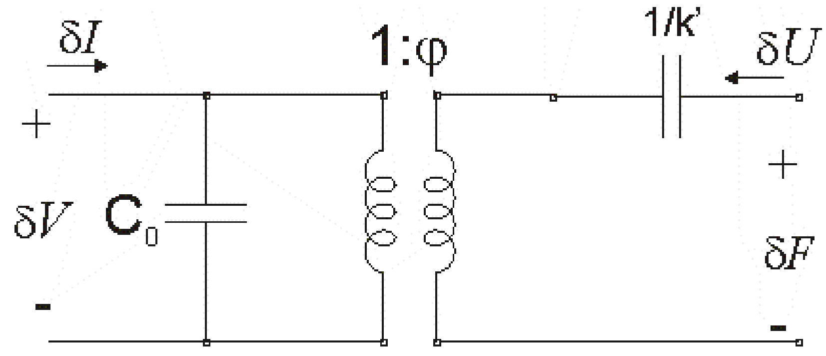
> *k'* represents the *effective* spring

> A combination of the mechanical spring *k* and the electrical spring

> This is an electrically tunable spring!

- Spring softening shows up in *k'*

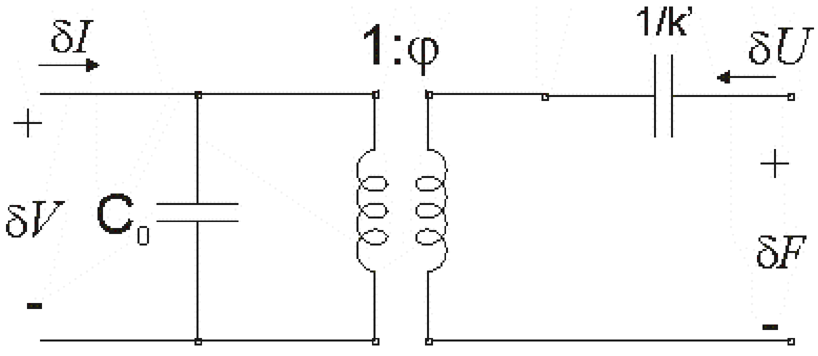
> As V_{in} increases, *k'* will decrease from *k* (at V_{in}=0) to 0 (at V_{in}=V_{pi})



$$k' = k - \frac{Q_0^2}{\varepsilon A \hat{g}_0}$$

Linearized Transducers

- φ **represents the electromechanical coupling**
- **Represents how much the capacitance changes with gap**
- **A measure of sensitivity**



$$\varphi = \frac{C_0 V_0}{\hat{g}_0} = \frac{Q_0}{\hat{g}_0}$$

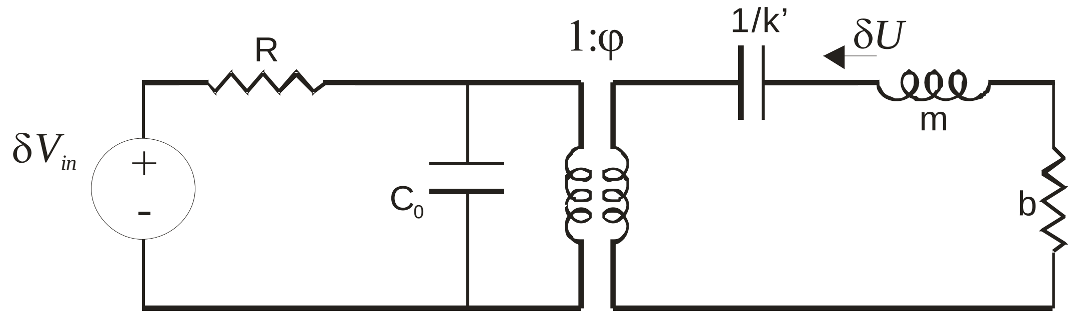
$$\boxed{\varphi = -V_0 \left.\frac{\partial C}{\partial g}\right|_{O.P.}} = -V_0 \frac{\partial}{\partial g} \left.\frac{\varepsilon A}{g}\right|_{O.P.}$$

$$= V_0 \frac{\varepsilon A}{\hat{g}_0^2}$$

$$= \frac{C_0 V_0}{\hat{g}_0}$$

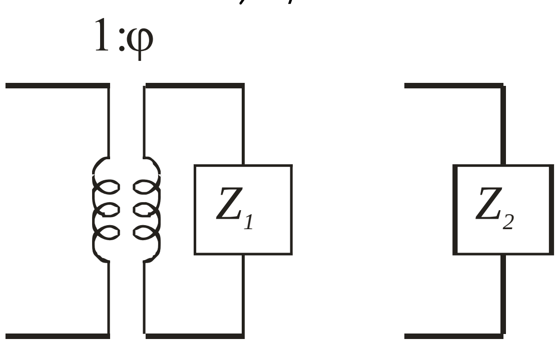
Transfer Functions

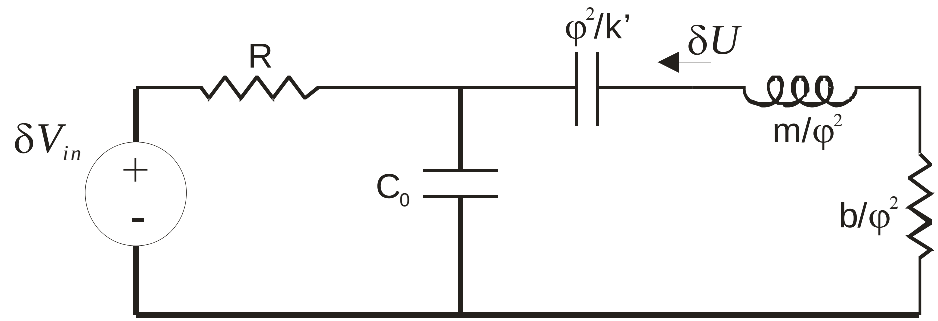
- Can use linearized circuit to construct H(s) using complex impedances
- Usually helpful to “eliminate” transformer



- Transformer changes impedances

$$Z_2 = Z_1 / \varphi^2$$





Can now get any transfer function using standard circuit analysis

Linearized Transducer Models

- Now we can understand Nguyen's filter!

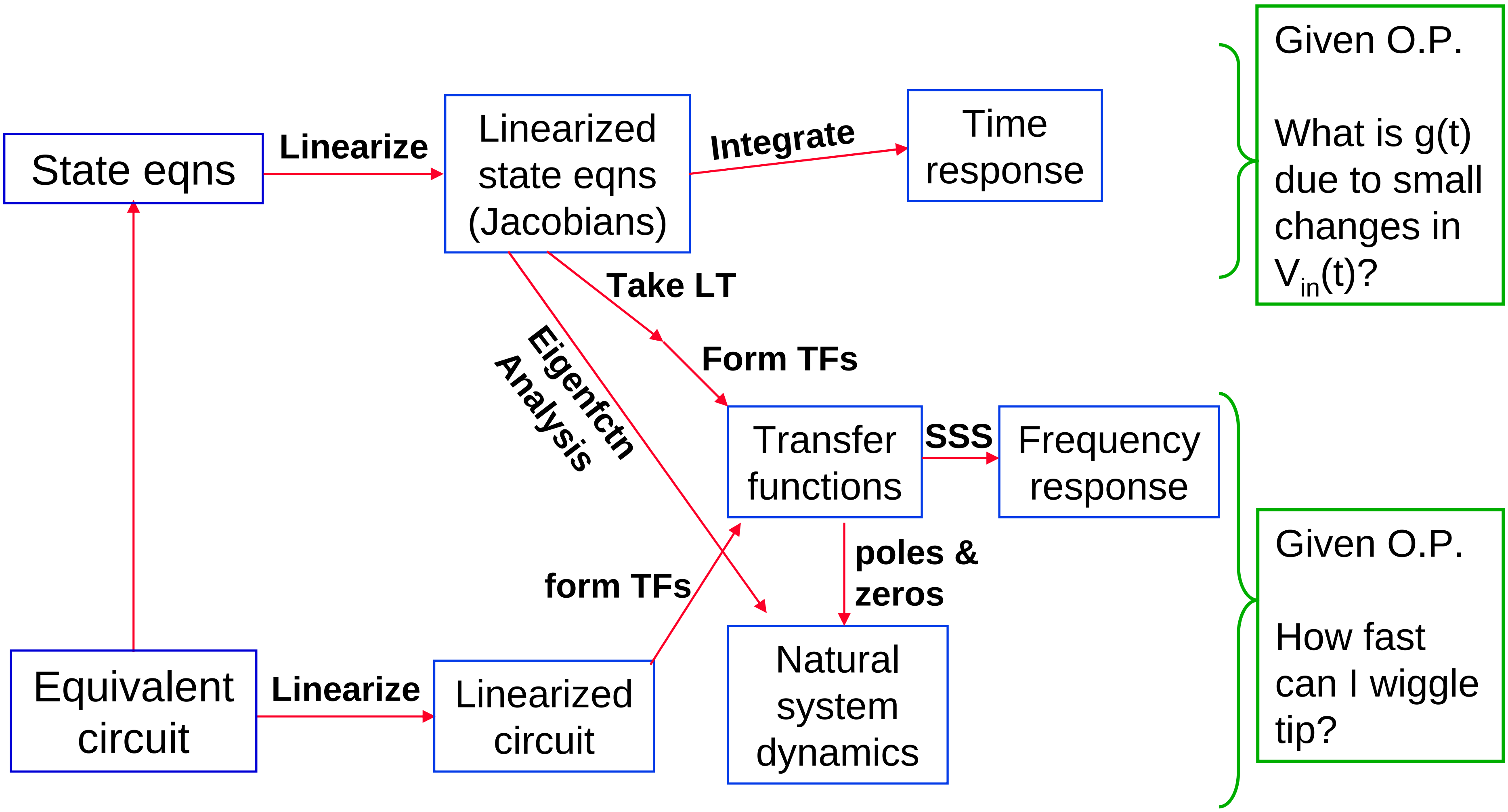
Image removed due to copyright restrictions.

Figure 9 on p. 17 in Nguyen, C. T.-C. "Vibrating RF MEMS Overview: Applications to Wireless Communications." *Proceedings of SPIE Int Soc Opt Eng* 5715 (January 2005): 11-25.

Image removed due to copyright restrictions.

Figure 12 on p. 62 in: Nguyen, C. T.-C. "Micromechanical Filters for Miniaturized Low-power Communications." *Proceedings of SPIE Int Soc Opt Eng* 3673 (July 1999): 55-66.

Small-signal analysis summary

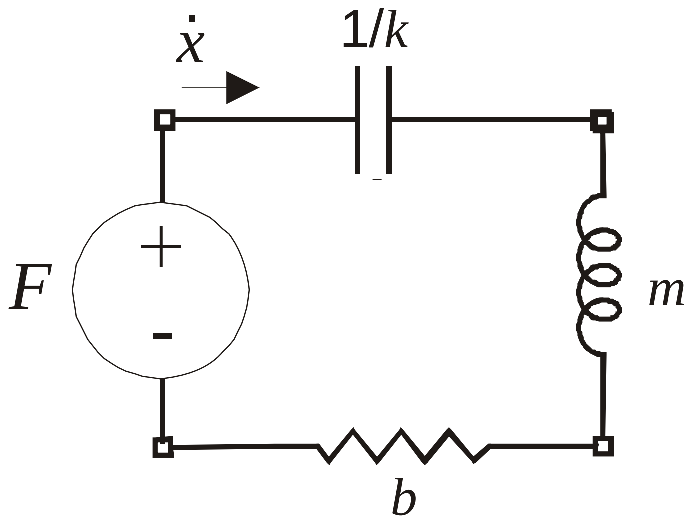


Conclusions

- **We can now analyze and design both quasistatic and dynamic behavior of our multi-domain MEMS**
- **We have much more powerful tools to analyze linear systems than nonlinear systems**
- **But most systems we encounter are nonlinear**
- **Linearization permits the study of small-signal inputs**
- **Next up: special topics in structures, heat transfer, fluids**

Review: analysis of a 2nd-order linear system

> **Spring-mass-dashpot**



$$\frac{d}{dt}\begin{bmatrix} x \\ \dot{x} \end{bmatrix} = \begin{bmatrix} \dot{x} \\ \frac{1}{m}\left(F - kx - b\dot{x}\right) \end{bmatrix}$$

State eqns ⇒

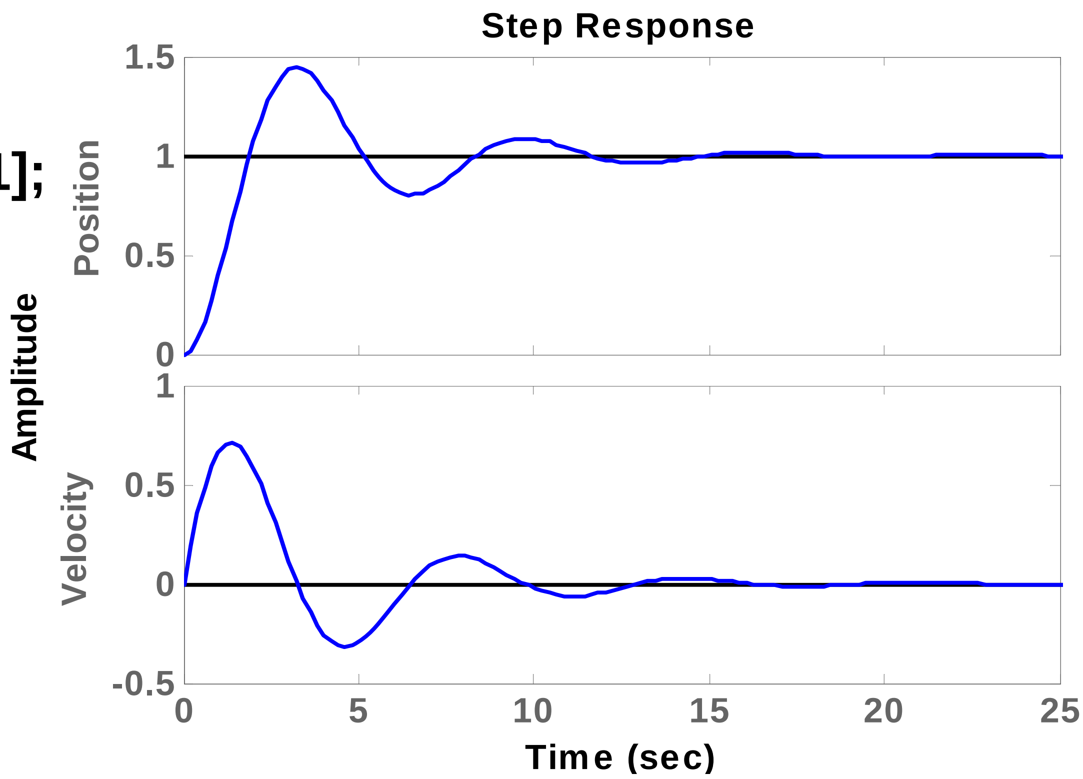
$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$
$$\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}$$

Direct Integration in Time

> **Example: Spring-mass-dashpot step response**

- ***k*=*m*=1;*b*=0.5;**

```
>> A=[0 1;-1 -0.5]; B=[0;1];
>> C=[1 0;0 1]; D=[0;0];
>> sys=ss(A,B,C,D);
>> step(sys)
```



Transfer Functions

- Can get TFs from A,B,C matrices

$$\mathbf{Y}(s) = \mathbf{CX}(s) + \mathbf{DU}(s)$$

$$\mathbf{Y}(s) = \mathbf{C}\left[(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{x}(0) + (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{BU}(s)\right] + \mathbf{DU}(s)$$

Assume transient has died out (X_{ZIR}=0)
No feed-through (D=0)

$$\mathbf{Y}(s) = \left[\mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\right]\mathbf{U}(s)$$

$$\mathbf{Y}(s) = \mathbf{H}(s)\mathbf{U}(s)$$

$$\mathbf{H}(s) = \left[\mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\right]$$

Transfer Functions

> **Let's do analytically & via MATLAB**

$$s\mathbf{I}-\mathbf{A}=\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix}-\begin{bmatrix} 0 & 1 \\ -k/m & -b/m \end{bmatrix}$$

$$=\begin{bmatrix} s & -1 \\ k/m & s+b/m \end{bmatrix}$$

$$(s\mathbf{I}-\mathbf{A})^{-1}=\frac{1}{\Delta}\begin{bmatrix} s+b/m & 1 \\ -k/m & s \end{bmatrix}$$

$$\Delta = s(s+b/m)+k/m = s^2+s\,b/m+k/m$$

$$\mathbf{C}(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{B}=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\frac{1}{\Delta}\begin{bmatrix} s+b/m & 1 \\ -k/m & s \end{bmatrix}\begin{bmatrix} 0 \\ 1/m \end{bmatrix}$$

$$=\frac{1}{\Delta}\begin{bmatrix} 1/m \\ s/m \end{bmatrix}$$

$$\mathbf{H}(s)=\begin{bmatrix} \frac{X(s)}{F(s)} \\ \frac{\dot{X}(s)}{F(s)} \end{bmatrix}=\begin{bmatrix} \frac{1}{ms^2+sb+k} \\ \frac{s}{ms^2+sb+k} \end{bmatrix}$$

$$\mathbf{H}(s)=\begin{bmatrix} \frac{1}{s^2+0.5s+1} \\ \frac{s}{s^2+0.5s+1} \end{bmatrix}$$

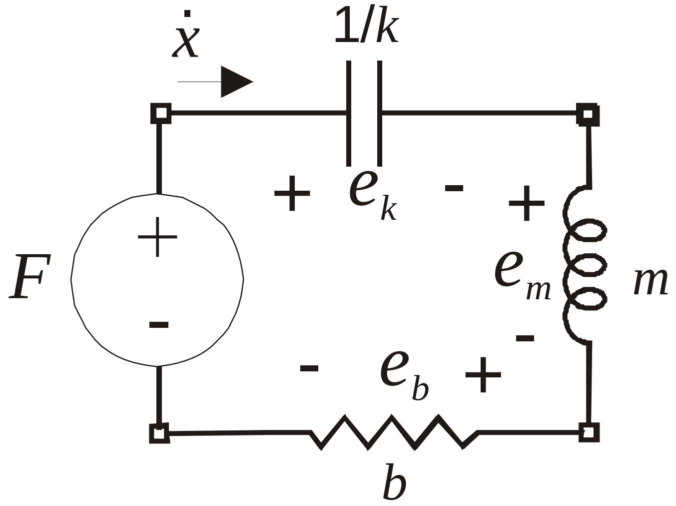
```
>> [n,d]=ss2tf(A,B,C,D)

n =     s2      s1         s0
        0   -0.0000    1.0000
        0    1.0000   -0.0000

d =
    1.0000    0.5000    1.0000
```

Transfer Functions

> **Can also construct H(s) directly using complex impedances and circuit model**



$$F - e_k - e_m - e_b = 0$$

$$e_k = \mathrm{k}x = \frac{\mathrm{k}}{s}\dot{x}$$

$$e_b = \mathrm{b}\dot{x}$$

$$e_m = \mathrm{m}\ddot{x} = \mathbf{m}s\dot{x}$$

$$\frac{\dot{\mathrm{X}}(s)}{\mathrm{F}(s)} = \mathrm{H}_2(s) = \frac{1}{\mathrm{Z}(s)}$$

$$= \frac{1}{\mathbf{b} + \mathbf{m}s + \mathbf{k}/s}$$

$$= \frac{s}{\mathbf{m}s^2 + \mathbf{b}s + \mathbf{k}}$$

$$\frac{\dot{\mathrm{X}}(s)}{\mathrm{F}(s)} = \frac{s\mathrm{X}(s)}{\mathrm{F}(s)}$$

$$\frac{\mathrm{X}(s)}{\mathrm{F}(s)} = \mathrm{H}_1(s) = \frac{1}{\mathbf{m}s^2 + \mathbf{b}s + \mathbf{k}}$$

Poles and Zeros

> **For 2nd-order system, easy to get poles and zeros from TFs**

$$\mathbf{H}(s) = \frac{1}{m}\begin{bmatrix} \dfrac{1}{s^2 + s\,b/m + k/m} \\ \dfrac{s}{s^2 + s\,b/m + k/m} \end{bmatrix}$$

$$= \frac{1}{m}\begin{bmatrix} \dfrac{1}{(s-s_1)(s-s_2)} \\ \dfrac{s}{(s-s_1)(s-s_2)} \end{bmatrix}$$

where

$$s_{1,2} = -\frac{b}{2m} \pm \sqrt{\left(\frac{b}{2m}\right)^2 - \frac{k}{m}}$$

these are the poles

Spring-mass-dashpot system

- **It is a second order system, with two poles**

$$s^2 + {}^{b}\!/_{m}\, s + {}^{k}\!/_{m} = s^2 + 2\alpha s + \omega_0^2$$

- **We conventionally define**
  - **Undamped resonant frequency**
  - **Damping constant**
  - **Damped resonant frequency**
  - **Quality factor**

$$\omega_0 = \sqrt{\frac{k}{m}}$$

$$\alpha = \frac{b}{2m}$$

$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$$

For underdamped systems $(\alpha < \omega_0)$

$$s_{1,2} = -\alpha \pm j\omega_{\mathrm{d}}$$

where

$$\omega_{\mathrm{d}} = \sqrt{\omega_0^{2-}\alpha^2}$$

Quality factor:

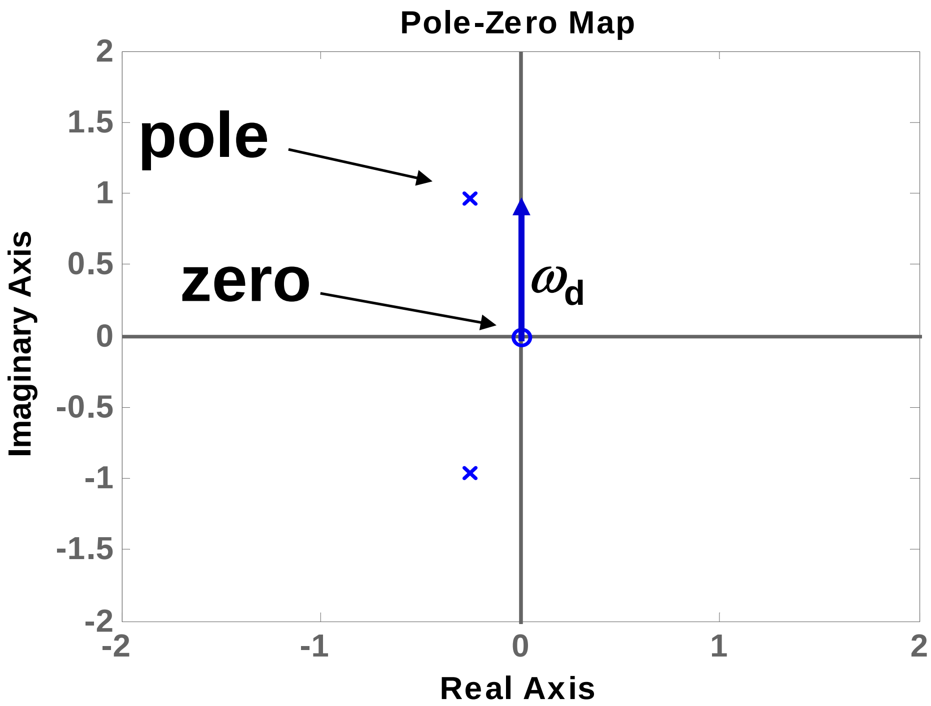
$$Q = \frac{\omega_0}{2\alpha} = \frac{m\omega_0}{b}$$

Pole-zero diagram

> **Displays information about dynamics of system function**

$$H_2(s) = \frac{s}{s^2 + 0.5s + 1}$$

$$s_{1,2} = -0.25 \pm j\sqrt{1 - \tfrac{1}{16}} = -0.25 \pm j0.97$$

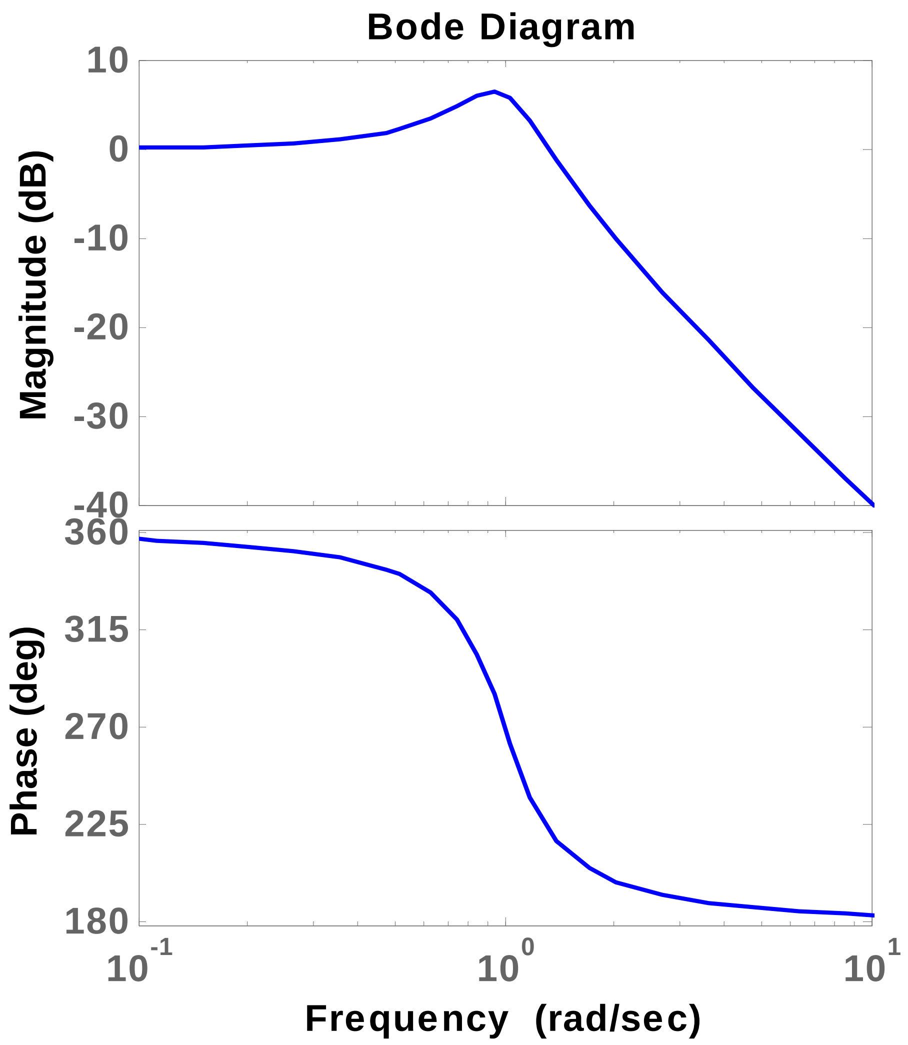


SMD-position frequency response

$$\mathbf{H}(j\omega) = \frac{1}{m}\frac{1}{-\omega^2 + 2\alpha j\omega + \omega_0^2}$$

$$\left|\mathbf{H}(j\omega)\right| = \frac{1}{m}\frac{1}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\alpha^2\omega^2}}$$

$$\angle\mathbf{H}(j\omega) = -\mathrm{atan}\left(\frac{2\alpha\omega}{\omega_0^2 - \omega^2}\right)$$



Eigenfunction Analysis

> **Find eigenvalues numerically using MATLAB and A matrix**

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 0 & -0.5 \end{bmatrix}$$

$$[\mathbf{V}, \Lambda] = \mathrm{eig}(\mathbf{A})$$

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} -0.25 + 0.97j & 0 \\ 0 & -0.25 - 0.97j \end{bmatrix}$$

$$\mathbf{V} = \begin{bmatrix} v_1 & v_2 \end{bmatrix} = \begin{bmatrix} 0.707 & 0.707 \\ -0.18 - 0.68j & -0.18 - 0.68j \end{bmatrix}$$