6.777J/2.372J Design and Fabrication of Microelectromechanical Devices
Spring Term 2007

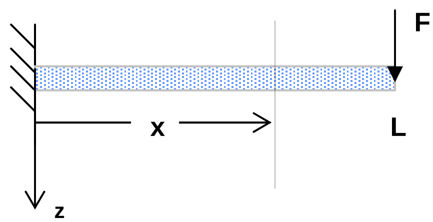
Massachusetts Institute of Technology

PROBLEM SET 3 (13 pts)

By Brian Taff (with acknowledgement to Feras Eid and Xue'en Yang) Due: Lecture 9

Problem 9.14 (2 pts): Bending of an AFM Cantilever

For LPCVD stoichiometric silicon nitride, E = 270 GPa.
From the lecture on structures, for a cantilever with the following configuration and loading



the differential equation of beam bending is: $\frac{d^2w}{dx^2} = -\frac{M}{EI}$

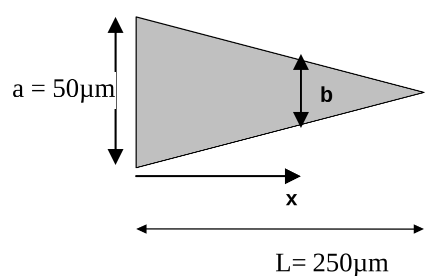
where M is the internal moment at distance x: $M = -F(L-x)$

and I is the moment of inertia of the cross section at the distance x. For a rectangular cross section:

$$I = bh^3/12$$

For our case, $h = 0.5$ μm and b is a function of x due to the triangular shape of the cantilever.

For our geometry: $$\frac{b}{L-x} = \frac{a}{L}$$



Therefore: $$I = \frac{a(L-x)h^3}{12L}$$

and $$\frac{d^2w}{dx^2} = \frac{12F(L-x)L}{Ea(L-x)h^3} = \frac{12FL}{Eah^3}$$

Integrating twice: $$w = \frac{6FL}{Eah^3}x^2 + Ax + B$$

Applying the boundary conditions: $w(x=0) = dw/dx\,(x=0) = 0$, we get: $A = B = 0$.

Therefore: $$w(x) = \frac{6FL}{Eah^3}x^2$$

At the tip ($x = L$), the deflection is: $$w(L) = \frac{6FL^3}{Eah^3} = 55.55\,F \quad [w \text{ in m}, F \text{ in } N]$$

And the equivalent spring constant is: $$k_{eq} = \frac{F}{w(L)} = \frac{Eah^3}{6L^3} = 0.018\ N/m$$

For a tip deflection of 2 μm: $$F = k_{eq} \cdot w(L) = 0.018 \cdot 2\text{x}10^{-6} = 3.6\text{x}10^{-8}\ N$$

And the maximum stress at the support is: $$\sigma_{\max} = \frac{Mh}{2I} = \frac{F(L-x)h12L}{2ah^3(L-x)} = \frac{6FL}{ah^2} = 4.32\ MPa$$

Problem 14.12 (2 pts): Circuit loading

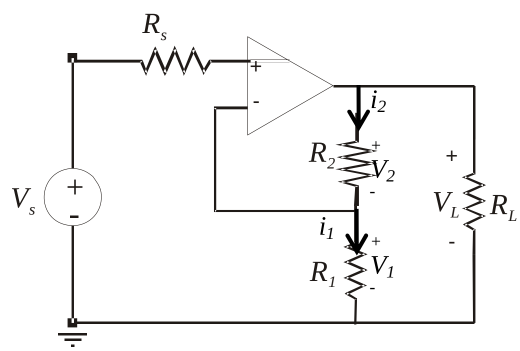
a) Assuming we have an ideal op-amp

$i^+ = 0 \rightarrow V^+ = V_s$
$i^- = 0 \rightarrow V^- = V_1,\ i_1 = i_2$

and $V^+ = V^- \rightarrow V_s = V_1$

Also $V_1 = i_1 R_1$, $V_L = i_1 (R_1 + R_2)$

Therefore $V_L / V_s = 1 + R_2 / R_1$



b) Assuming again we have an ideal op-amp

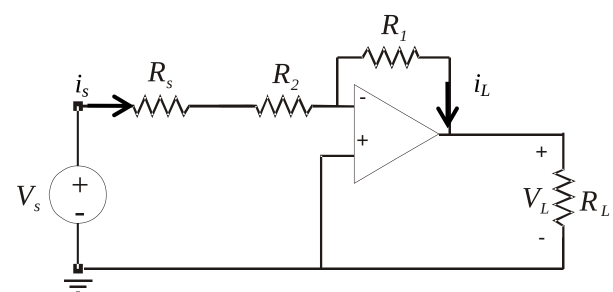
$i^- = 0 \rightarrow i_s = i_L$

and $V^+ = V^- = 0$

Also $V^- - V_L = i_L R_1$, $V_s - V^- = i_s (R_s + R_2)$

Therefore $V_L / V_s = -R_1 / (R_s + R_2)$

The ratio is negative, hence the name "inverting".



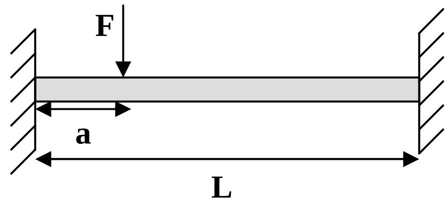
c) The gain G is defined as the ratio of the output voltage to the input voltage, or V_L / V_s.

For the non-inverting amplifier: $G_{ideal} = 1 + R_2 / R_1 = G_{actual}$ for all values of R_s
$G_{actual} / G_{ideal} = 1$

For the inverting amplifier: $G_{ideal} = -R_1 / R_2$
$G_{actual} / G_{ideal} = R_2 / (R_s + R_2)$

Thus for the non-inverting amplifier, the ideal and actual gains are always equal, while for the inverting amplifier, the actual gain approaches the ideal gain only when the sensor impedance approaches zero (assuming both amplifiers are ideal). Clearly, the non-inverting amplifier is less sensitive to changes in R_s than the inverting one.

Problem 9.13 (3pts): Leveraged Bending with Electrostatic Actuation



a) The governing equation for the deflection of a doubly clamped beam under a point load as shown in the figure is,

$$EI\frac{d^4w}{dx^4} = -F < x-a >^{-1},$$

where $< x-a >^{-1}$ denotes a concentrated load at a. Integrating, we have,

$$EI\frac{d^3w}{dx^3} = -F < x-a >^{0} + C_1$$

$$EI\frac{d^2w}{dx^2} = -F < x-a >^{1} + C_1x + C_2$$

$$EI\frac{dw}{dx} = -\frac{F}{2} < x-a >^{2} + \frac{1}{2}C_1x^2 + C_2x + C_3$$

$$EIw = -\frac{F}{6} < x-a >^{3} + \frac{1}{6}C_1x^3 + \frac{1}{2}C_2x^2 + C_3x + C_4$$

Apply the boundary conditions

$$w(0) = w(L) = w'(0) = w'(L) = 0\ ,$$

we get,

$$C_1 = \frac{F(L-a)^2(L+2a)}{L^3}$$

$$C_2 = \frac{-F(L-a)^2a}{L^2}$$

$$C_3 = C_4 = 0$$

Hence, the center deflection is,

$$w\left(\frac{L}{2}\right) = \frac{Fa^2(4a-3L)}{48EI}$$

If we next examine the effect of the point load acting at the right hand side of the beam (acting alone as if the load at "a" had been removed), we get a similar setup.

$$EI\frac{d^4w}{dx^4} = -F < x-(L-b) >^{-1}$$

$$EI\frac{d^3w}{dx^3} = -F < x-(L-b) >^{0} + C_1$$

$$EI\frac{d^2w}{dx^2} = -F < x-(L-b) >^{1} + C_1x + C_2$$

$$EI\frac{dw}{dx} = -\frac{F}{2} < x-(L-b) >^{2} + \frac{1}{2}C_1x^2 + C_2x + C_3$$

$$EIw = -\frac{F}{6} < x-(L-b) >^{3} + \frac{1}{6}C_1x^3 + \frac{1}{2}C_2x^2 + C_3x + C_4$$

Applying the boundary conditions

$$w(0) = w(L) = w'(0) = w'(L) = 0\ ,$$

we get,

$$C_1 = \frac{Fb^2(3L-2b)}{L^3}$$
$$C_2 = \frac{Fb^2(b-L)}{L^2}$$
$$C_3 = C_4 = 0$$

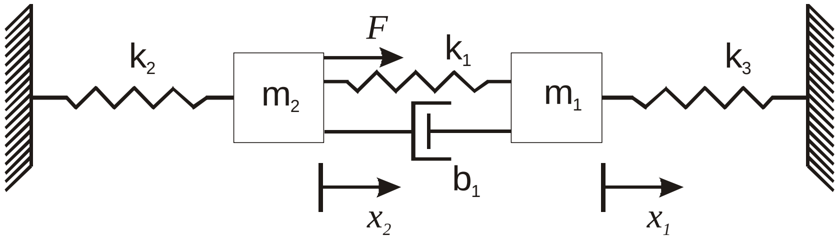
In this case, the center deflection is:

$$w\left(\frac{L}{2}\right) = \frac{Fb^2(4b-3L)}{48EI}$$

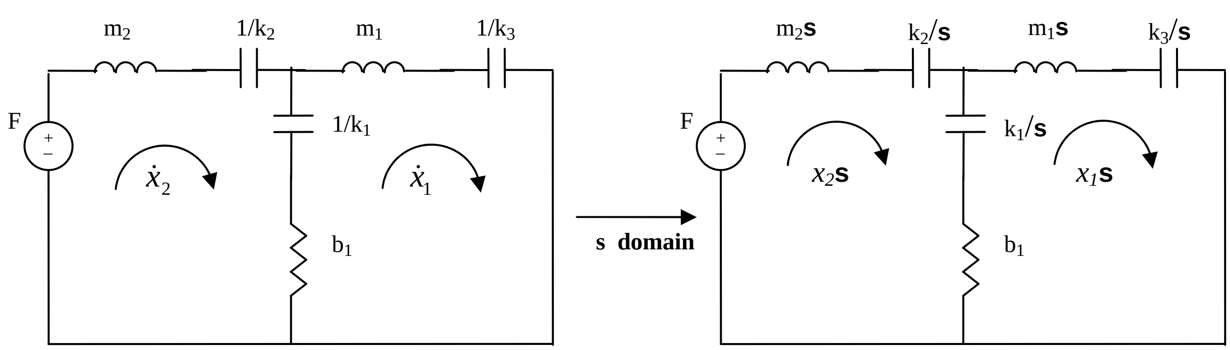
Applying superposition to the beam, we can obtain the center deflection due to both loads to be,

$$\delta = \frac{F\left[a^2(4a-3L)+b^2(4b-3L)\right]}{48EI}$$

Problem 5.9 (2 pts): Circuit representation of a lumped mechanical system



In the $e \rightarrow V$ convention, the flows are the time derivatives of the displacements x_i of the different bodies, and effort is the external force. Since k_1 and b_1 share the same displacement, and hence, the same flow, they are in series. The same holds for k_2 and m_2 and also for k_3 and m_1. By observation, the net flow through b_1 is $(\dot{x}_2 - \dot{x}_1)$. Hence, the equivalent circuit can be represented as shown in Figure.



Apply KVL for the left loop, we have

$$m_2 s^2 x_2 + k_2 x_2 + (k_1/s + b_1)(x_2 s - x_1 s) = F$$
$$\Rightarrow -(b_1 s + k_1)x_1 + (m_2 s^2 + b_1 s + k_1 + k_2)x_2 = F \qquad (1)$$

Apply KVL for the right loop, we have

$$m_1 s^2 x_1 + k_3 x_1 + (k_1 / s + b_1)(x_1 s - x_2 s) = 0$$
$$\Rightarrow (m_1 s^2 + b_1 s + k_1 + k_3) x_1 - (b_1 s + k_1) x_2 = 0$$
$$\Rightarrow x_2 = \frac{m_1 s^2 + b_1 s + k_1 + k_3}{b_1 s + k_1} x_1$$

Substitute into (1), we have,

$$\frac{x_1(s)}{F(s)} = \frac{b_1 s + k_1}{-(b_1 s + k_1)^2 + (m_1 s^2 + b_1 s + k_1 + k_3)(m_2 s^2 + b_1 s + k_1 + k_2)}$$