6.730 Physics for Solid State Applications

Lecture 4: Vibrations in Solids

Outline

- Review Lecture 3
- Sommerfeld Theory of Metals
- 1-D Elastic Continuum
- 1-D Lattice Waves
- 3-D Elastic Continuum
- 3-D Lattice Waves

Microscopic Variables for Electrical Transport

Drude Theory

Balance equation for forces on electrons:

$$m\frac{d\mathbf{v}(r,t)}{dt} = -\underbrace{m\frac{\mathbf{v}(r,t)}{\tau}}_{\text{DRAG FORCE}} \underbrace{-e\left[\mathbf{E}(\mathbf{r},\mathbf{t}) + \mathbf{v}(\mathbf{r},\mathbf{t}) \times \mathbf{B}(\mathbf{r},\mathbf{t})\right]}_{\text{LORENTZ FORCE}}$$

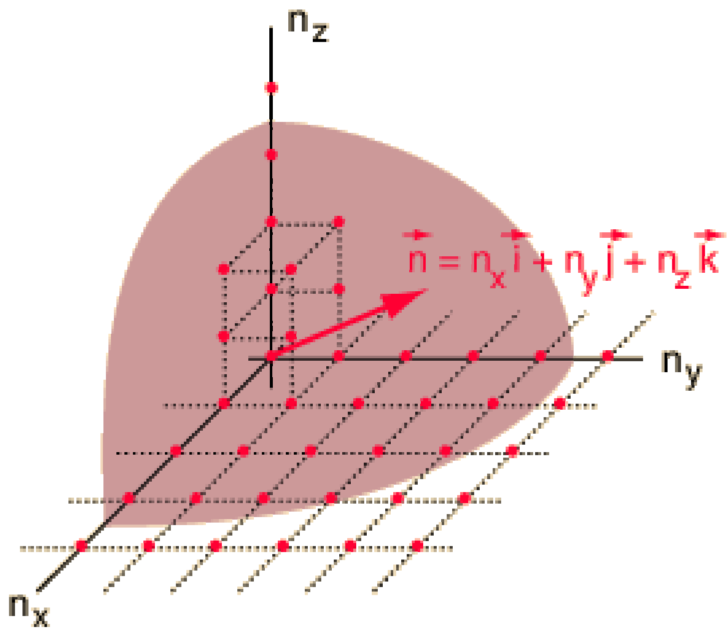
In steady-state when **B**=0:

$$\mathbf{v} = -\frac{e\tau}{m}\mathbf{E}_{\text{DC}}$$

$$\mathbf{J} = -ne\mathbf{v} = \frac{ne^2\tau}{m}\mathbf{E}_{\text{DC}}$$

$$\boxed{\mathbf{J} = \sigma\mathbf{E}_{\text{DC}} \quad \text{and} \quad \sigma = \frac{\mathbf{ne^2}\tau}{\mathbf{m}}}$$

Density of States



$$n = \frac{N}{V} = \int_{-\infty}^{\infty} \frac{1}{1 + e^{(E_{\mathbf{k}} - \mu)/k_B T}} 2 \frac{d^3\mathbf{k}}{(2\pi)^3}$$

$$n = \int_{-\infty}^{\infty} g(E) f(E - \mu) dE = \int_{-\infty}^{\infty} g(E) \frac{1}{1 + e^{(E - \mu)/k_B T}} dE$$

Microscopic Variables for Electrical Transport

Balance equation for energy of electrons:

$$\frac{dE}{dt} = -\frac{\Delta E}{\tau} + IV$$

In steady-state:

$$\Delta E = \tau IV$$

In the continuum models, we assume that electron scattering is sufficiently fast that all the energy pumped into the electrons is randomized; all additional energy heats the electrons

How do we relate ΔE and T ?

Specific Heat and Heat Capacity

Again assume that the heat and change in internal energy are the same:

$$c_V = \left(\frac{dQ}{dT}\right)_V = \left(\frac{dE_{\text{total}}}{dT}\right)_V \qquad \text{(heat capacity)}$$

Take constant volume since this ensures
none of the extra energy is going into *work*
(think ideal gas)

$$C_V = \frac{1}{V}\frac{d}{dT}\left(\frac{3}{2}Nk_BT\right) = \frac{3}{2}nk_B \qquad \text{(specific heat)}$$

$$C_v = 2\times 10^6 \frac{\text{erg}}{\text{cm}^3\text{-K}} = 11\frac{\text{Joule}}{\text{mole-K}}$$

Specific heat is independent of temperature…Law of Dulong and Petit

Specific Heat Measurements

(hyperphysics.phy-astr.gsu.edu)

Specific heat is independent of temperature…NOT TRUE !

To get this correct we will need to (a) quantize electron energy levels, (b) introduce discreteness of lattice and (c) the heat capacity of lattice

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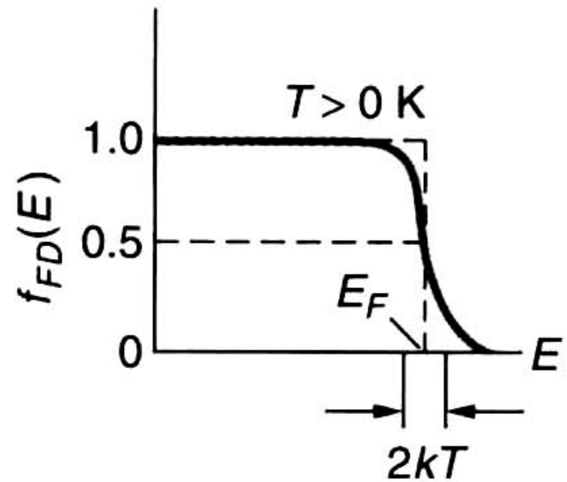
Low Temperature Specific Heat of the Free Electron Gas

Sommerfeld Approximation



$$\frac{\Delta E}{V} \approx \underbrace{[g(E_{Fo})k_B T]}_{\text{excited states}} \qquad \underbrace{k_B T}_{\text{increase in energy}}$$

$$C_V = \frac{\partial(\Delta E/\mathsf{V})}{\partial T} \approx 2g(E_{F0})k_B^2 T$$

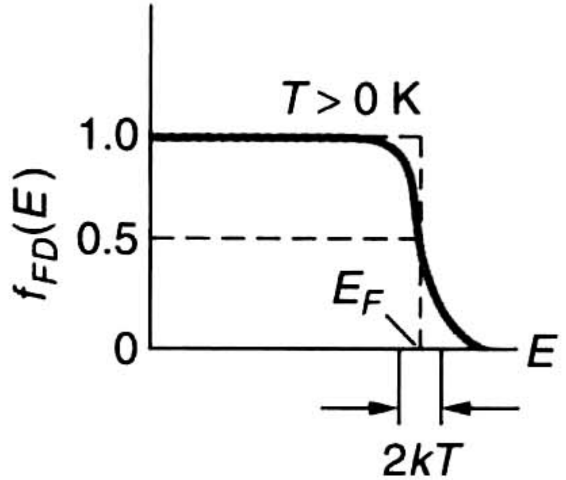
$$C_V \approx \left(\frac{3}{2}nk_B\right)\left(\frac{2k_BT}{E_{F0}}\right) \approx \left(\frac{3}{2}nk_B\right)(100 - 5000)$$

Conductivity of the Free Electron Gas
Sommerfeld Approximation

$$\mathbf{v_d} = (-\mathbf{e}\tau/\mathbf{m})\mathbf{E}_{\mathrm{DC}}$$

$$\mathbf{v} = \mathbf{v_F} - \frac{\mathbf{e}\tau}{\mathbf{m}}\mathbf{E}_{\mathrm{DC}}$$

$$E = \frac{1}{2}mv^2 \approx \frac{1}{2}mv_F^2 + e\tau\mathbf{v}\cdot\mathbf{E}_{\mathrm{DC}}$$



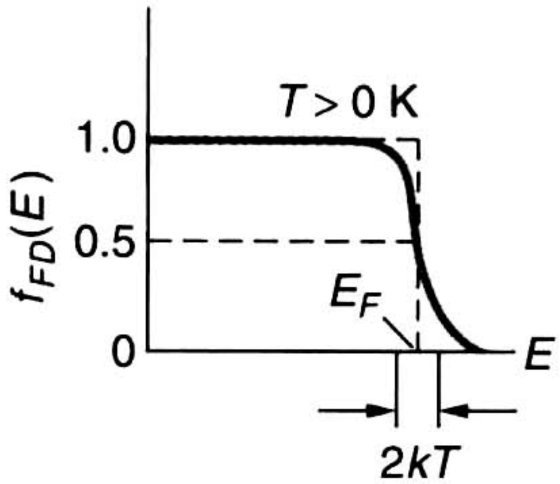
$$\Delta E = e\tau v_F|\mathbf{E}_{\mathrm{DC}}|$$

$$J = -e(\delta n)v_F$$

$$\delta n \approx g(E_F)\Delta E$$

Only electrons near E_F contribute to current !

Conductivity of the Free Electron Gas
Sommerfeld Approximation



$$\Delta E = e\tau v_F |\mathbf{E}_{\mathrm{DC}}|$$

$$J = -e(\delta n) v_F$$

$$\delta n \approx g(E_F)\Delta E$$

$$J = e^2 \tau v_F^2 g(E_F) |\mathbf{E}_{\mathrm{DC}}|$$

$$\sigma = e^2 v_F^2 \tau g(E_F)$$

$$\sigma \approx \frac{ne^2\tau}{m}$$

Sommerfeld recovers the phenomenological results !

Sommerfeld Expansion

$$f(E-\mu) = \lim_{T\to 0} \frac{1}{1+e^{(E-\mu)/k_B T}} = 1-u(E-\mu)$$

$$f'(E-\mu) = -\delta(E-E_{Fo})$$

$$\int_{-\infty}^{\infty} f(E-\mu)H(E)dE = \int_{-\infty}^{\mu} H(E)dE + \frac{\pi^2}{6}(k_B T)^2 H'(\mu) + O\left(\frac{k_B T}{E_{F0}}\right)^4$$

$$\int_{-\infty}^{\mu} H(E)dE = \int_{-\infty}^{E_{F0}} H(E)dE + (\mu - E_{F0})H(E_{F0}) + O\left(\frac{k_B T}{E_{F0}}\right)^4$$

Sommerfeld Expansion for Electron Density

$$n = \underbrace{\int_0^{E_{F0}} g(E)dE}_{\approx n} + \underbrace{\left[(\mu - E_{F0})g(E_{F0}) + \frac{\pi^2}{6}(k_B T)^2 g'(E_{F0})\right]}_{0}$$

$$\mu = E_{F0}\left\{1 - \frac{\pi^2}{6}\left(\frac{(k_B T)^2}{E_{F0}}\right)\frac{g'(E_{F0})}{g(E_{F0})}\right\}$$

$$\mu = E_{F0}\left\{1 - \frac{1}{3}\left(\frac{\pi k_B T}{2E_{F0}}\right)^2\right\}$$

Sommerfeld Expansion for Electron Energy

$$\frac{E}{V} = \int_{-\infty}^{\infty} E g(E) f(E-\mu) dE$$

$$= \int_{0}^{E_{F0}} E g(E) dE + E_{F0} \underbrace{\left[(\mu - E_{F0}) g(E_{F0}) + \frac{\pi^2}{6} (k_B T)^2 g'(E_{F0}) \right]}_{0}$$

$$+ \frac{\pi^2}{6} (k_B T)^2 g(E_{F0}) + O(T^4)$$

$$\frac{E}{V} = \int_{0}^{E_{F0}} E g(E) dE + \frac{\pi^2}{6} (k_B T)^2 g(E_{F0})$$

$$= \frac{3}{5} E_F n + \frac{\pi^2}{6} (k_B T)^2 g(E_{F0})$$

$$\boxed{C_V = \left. \frac{\partial\left((E/V)\right)}{\partial T} \right|_{V,N} = \frac{\pi^2}{3} k_B^2 T g(E_{F0}) = \gamma T}$$

Specific Heat Measurements

(hyperphysics.phy-astr.gsu.edu)

To get this correct we will need to (a) quantize electron energy levels, (b) introduce discreteness of lattice and (c) the heat capacity of lattice

Density of States is the Central Character in this Story

Goal: Calculate electrical properties (eg. resistance) for solids

Approach:
In the end calculating resistance boils down to calculating the electronic energy levels and wavefunctions; to knowing the *bandstructure*

You will be able to relate a bandstructure to macroscopic parameters for the solid

$$\sigma = e^2 v_F^2 \tau g(E_F)$$

$$C_V = \left.\frac{\partial\left(\left(E/V\right)\right)}{\partial T}\right|_{V,N} = \frac{\pi^2}{3} k_B^2 T g(E_{F0}) = \gamma T$$

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- 1-D Lattice Waves
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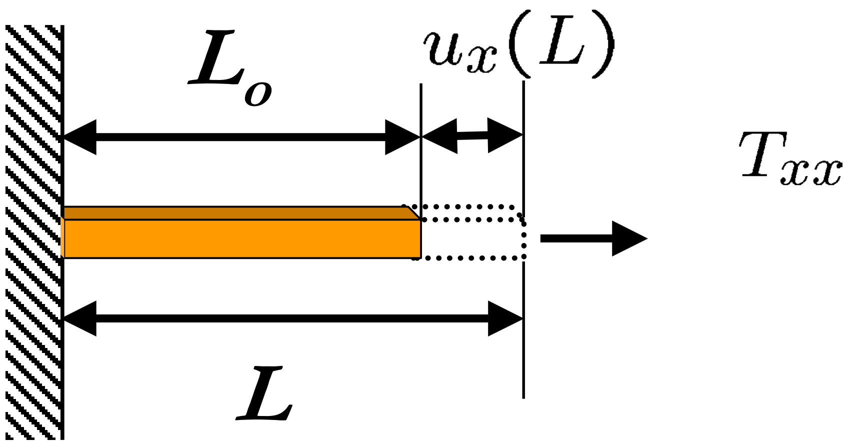
1-D Elastic Continuum
Stress and Strain

uniaxial loading



Stress:

$$T_{xx} = \frac{F_x}{A} \left[N/m^2 \right]$$

Strain:

$$\delta(dx) = u_x(x + dx) - u_x(x)$$

Normal strain:

$$E_{xx} = \frac{\delta(dx)}{dx} = \frac{\partial u_x}{\partial x}$$

If u_x is uniform there is no strain, just rigid body motion.

1-D Elastic Continuum

Young's Modulus

$$T_{xx} = E_Y E_{xx}$$

Young's Modulus For Various Materials (GPa)
from Christina Ortiz

CERAMICS GLASSES AND SEMICONDUCTORS

Diamond (C)	1000
Tungsten Carbide (WC)	450 -650
Silicon Carbide (SiC)	450
Aluminum Oxide (Al_2O_3)	390
Berylium Oxide (BeO)	380
Magnesium Oxide (MgO)	250
Zirconium Oxide (ZrO)	160 - 241
Mullite ($Al_6Si_2O_{13}$)	145
Silicon (Si)	107
Silica glass (SiO_2)	94
Soda-lime glass (Na_2O - SiO_2)	69

METALS :

Tungsten (W)	406
Chromium (Cr)	289
Berylium (Be)	200 - 289
Nickel (Ni)	214
Iron (Fe)	196
Low Alloy Steels	200 - 207
Stainless Steels	190 - 200
Cast Irons	170 - 190
Copper (Cu)	124
Titanium (Ti)	116
Brasses and Bronzes	103 - 124
Aluminum (Al)	69

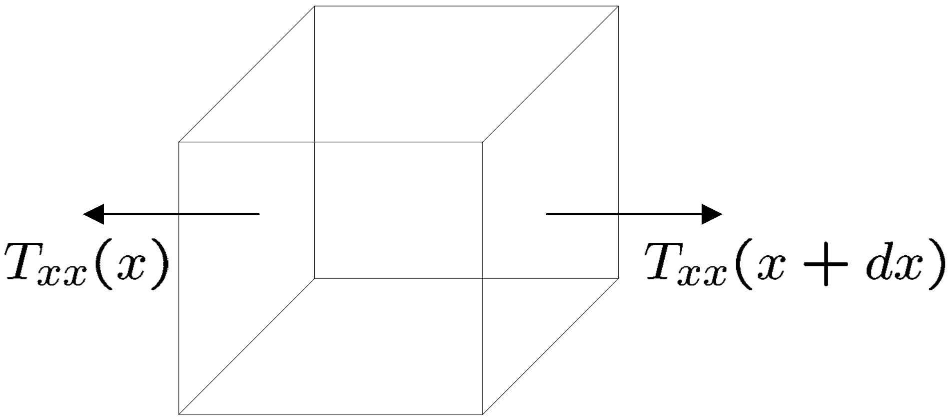
PINE WOOD (along grain): 10

POLYMERS :

Polyimides	3 - 5
Polyesters	1 - 5
Nylon	2 - 4
Polystryene	3 - 3.4
Polyethylene	0.2 -0.7
Rubbers / Biological Tissues	0.01-0.1

Dynamics of 1-D Continuum

1-D Wave Equation



Net force on incremental volume element:

$$f_x = [T_{xx}(x + dx) - T_{xx}(x)]\; dy\, dz$$

$$m\frac{\partial^2 u_x}{\partial t^2} = [T_{xx}(x + dx) - T_{xx}(x)]\; dy\, dz$$

$$\rho\frac{\partial^2 u_x}{\partial t^2}\, dx\, dy\, dz = [T_{xx}(x + dx) - T_{xx}(x)]\; dy\, dz$$

$$\boxed{\rho\frac{\partial^2 u_x}{\partial t^2} = \frac{\partial T_{xx}}{\partial x}}$$

Dynamics of 1-D Continuum

1-D Wave Equation

$$\rho \frac{\partial^2 u_x}{\partial t^2} = \frac{\partial T_{xx}}{\partial x} \qquad T_{xx} = E_Y E_{xx} \qquad E_{xx} = \frac{\partial u_x}{\partial x}$$

$$\rho \frac{\partial^2 u_x}{\partial t^2} = E_Y \frac{\partial^2 u_x}{\partial x^2}$$

$$\frac{\partial^2 u_x}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 u_x}{\partial t^2} \qquad c = \sqrt{\frac{E_Y}{\rho}}$$

Velocity of sound, c, is proportional to stiffness and inverse prop. to inertia

Dynamics of 1-D Continuum

1-D Wave Equation Solutions

$$\frac{\partial^2 u_x}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u_x}{\partial t^2}$$

Clamped Bar: Standing Waves

$$u_x(x,t) = A_{\pm} \sin(kx) \exp(i\omega t) \qquad \omega = ck$$

$$u_{x,m,\pm}(x,t) = A_{m,\pm} \sin\left(\frac{m\pi x}{L}\right) \exp\left(\pm i \frac{m\pi c}{L} t\right)$$

$$k = \frac{m\pi}{L} \qquad \text{for} \qquad m = 1, 2, \ldots$$

Dynamics of 1-D Continuum

1-D Wave Equation Solutions

$$\frac{\partial^2 u_x}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u_x}{\partial t^2}$$

Periodic Boundary Conditions: Traveling Waves

$$u_x(x,t) = A_{\pm}\,\mathrm{exp}(ikx)\,\mathrm{exp}\,(i\omega t) \qquad\qquad \omega = ck$$

$$u_{x,n,\pm}(x,t) = B_{n,\pm}\,\mathrm{exp}\left(\pm i\frac{2n\pi x}{L}(x \pm ct)\right)$$

$$k = \frac{2n\pi}{L} \qquad \text{for} \qquad n = \pm 1, \pm 2, \ldots$$

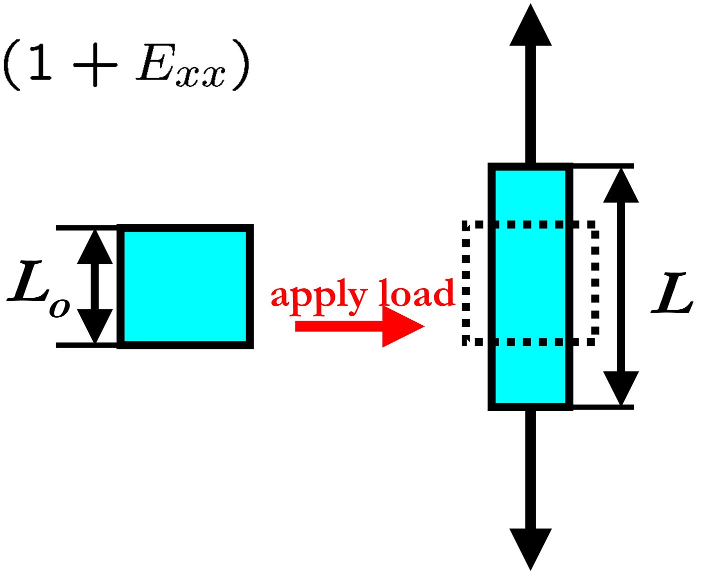
3-D Elastic Continuum

Volume Dilatation

$$dx \rightarrow dx + \delta(dx) = dx + E_{xx}dx = dx(1 + E_{xx})$$

$$dy \rightarrow dy(1 + E_{yy})$$

$$dz \rightarrow dz(1 + E_{zz})$$



$$e = \frac{\delta V}{V} = \frac{dx(1 + E_{xx})dy(1 + E_{yy})dz(1 + E_{zz}) - dxdydz}{dxdydz}$$

$$e = E_{xx} + E_{yy} + E_{zz}$$

Volume change is sum of all three normal strains

3-D Elastic Continuum
Poisson's Ratio

$$E_{xx} = \frac{\partial u_x}{\partial x} \qquad E_{yy} = \frac{\partial u_y}{\partial y} \qquad E_{zz} = \frac{\partial u_z}{\partial z}$$

$$e = E_{xx} + E_{yy} + E_{zz} = \nabla \cdot \mathbf{u}(\mathbf{r})$$

ν is Poisson's Ratio – ratio of lateral strain to axial strain

$$E_{yy} = E_{zz} = -\nu E_{xx}$$

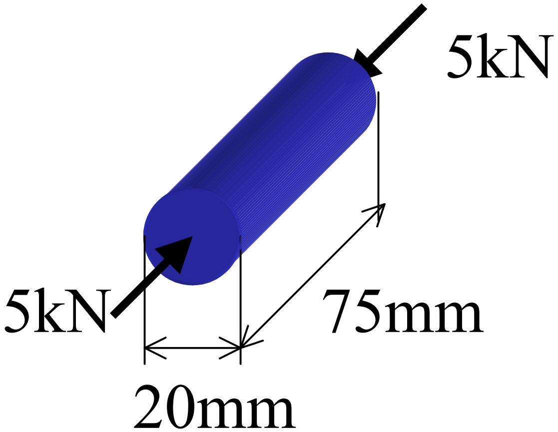
$$e = E_{xx}(1 - 2\nu)$$

Poisson's ratio can not exceed 0.5, typically 0.3

3-D Elastic Continuum

Poisson's Ratio Example

Aluminum: E_Y=68.9 GPa, ν = 0.35



3-D Elastic Continuum

Poisson's Ratio Example

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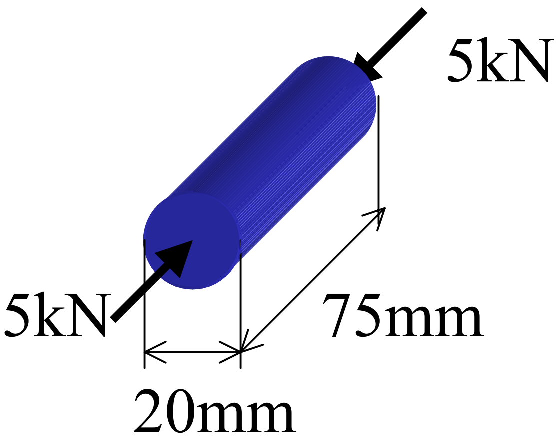


$$T_{xx} = \frac{F_x}{A} = \frac{5\text{x}10^3}{\pi(10\text{x}10^{-3})^2} = -15.9\text{MPa}$$

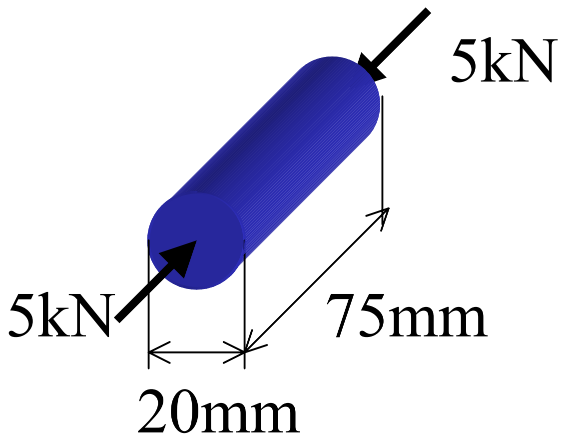
$$E_{xx} = \frac{T_{xx}}{E_Y} = \frac{-15.9\text{x}10^6}{68.9\text{x}10^9} = -0.231\text{x}10^{-3}$$

$$E_{xx} = \frac{\Delta l}{l} = -0.231\text{x}10^{-3}$$

$$\Delta l = -0.0173\text{mm}$$

3-D Elastic Continuum
Poisson's Ratio Example

Aluminum: E_Y=68.9 GPa, ν = 0.35



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$$E_{xx} = \frac{\Delta l}{l} = -0.231\text{x}10^{-3}$$

$$\Delta l = -0.0173\text{mm}$$

$$E_{trns} = -\nu E_{xx} = -0.35 E_{xx} = 0.081\text{x}10^{-3}$$

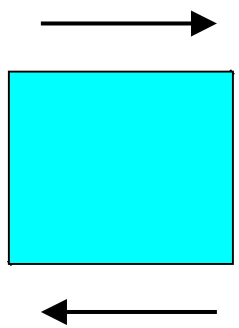
$$E_{trns} = \frac{\Delta d}{d}$$

$$\Delta d = +0.001617\text{mm}$$

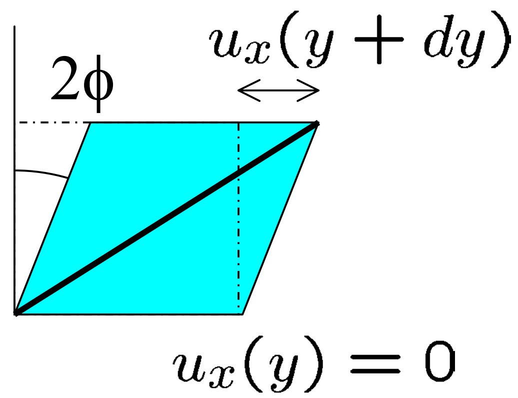
3-D Elastic Continuum

Shear Strain

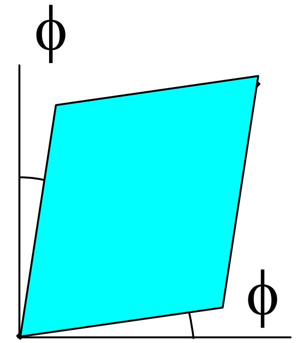
Shear loading



Shear plus rotation



Pure shear



Pure shear strain

$$\phi = E_{xy} = \frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right)$$

Shear stress

$$T_{xy} = G\,2\phi = 2GE_{xy}$$

G is shear modulus

3-D Elastic Continuum

Stress and Strain Tensors

$$E_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

$$E_{xx} = \frac{\partial u_x}{\partial x}$$

$$E_{xy} = \frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right)$$

$$e = \sum_{k=1}^{3} E_{kk}$$

For *most* general isotropic medium,

$$\mathbf{T} = \lambda \mathbf{e} \mathbf{I} + \mathbf{2}\mu \mathbf{E}$$

Initially we had three elastic constants: E_Y, G, e

Now reduced to only two: λ, μ

3-D Elastic Continuum
Stress and Strain Tensors

$$T_{ij} = \lambda e \, \delta_{ij} + 2\mu E_{ij}$$

If we look at just the diagonal elements

$$\sum_{k=1}^{3} T_{kk} = 3\lambda e + 2\mu e$$

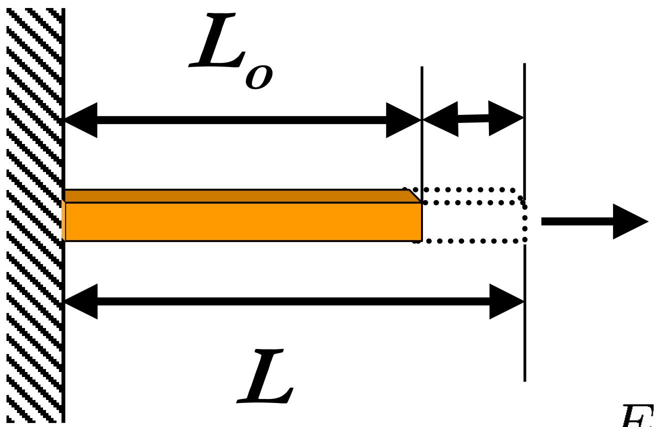
$$e = \frac{1}{3\lambda + 2\mu} \sum_{k=1}^{3} T_{kk}$$

Inversion of stress/strain relation:

$$E_{ij} = \frac{1}{2\mu} \left[T_{ij} - \frac{\lambda}{3\lambda + 2\mu} \left(\sum_{k} T_{kk} \right) \delta_{ij} \right]$$

3-D Elastic Continuum

Example of Uniaxial Stress



$T_{11} \neq 0$

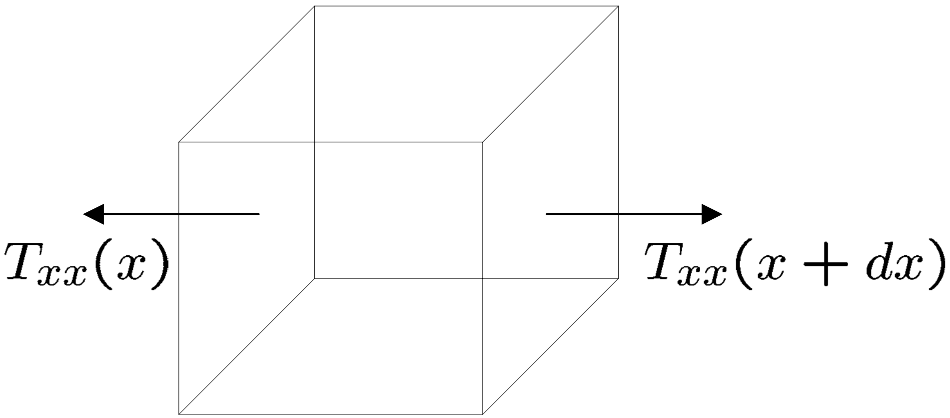
$$E_{ij} = \frac{1}{2\mu}\left[T_{ij} - \frac{\lambda}{3\lambda + 2\mu}\left(\sum_k T_{kk}\right)\delta_{ij}\right]$$

$$E_{11} = \underbrace{\frac{\lambda + \mu}{\mu(3\lambda + 2\mu)}}_{E_Y} T_{11}$$

$$E_{22} = E_{33} = -\underbrace{\frac{\lambda}{2(\lambda + \mu)}}_{\nu} E_{11}$$

Dynamics of 3-D Continuum

3-D Wave Equation



Net force on incremental volume element:

$$\mathbf{F} = \int_{\mathbf{V}} \mathbf{f}\,\mathbf{dxdydz}$$

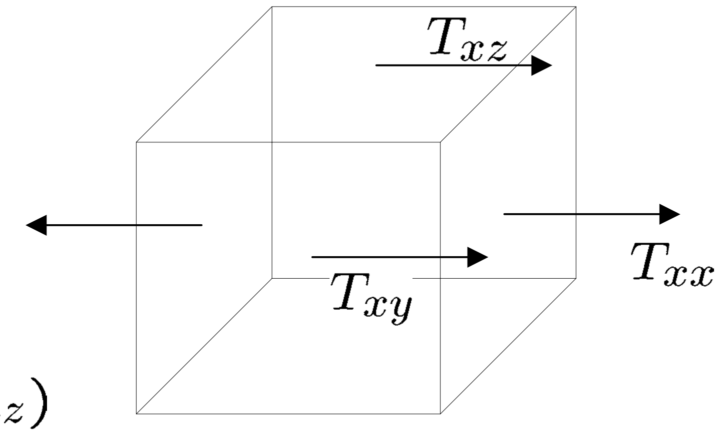
$$\mathbf{F} = \int_{\mathbf{v}} \rho \frac{\partial^2 \mathbf{u}}{\partial \mathbf{t}^2} \mathbf{dxdydz}$$

$$\mathbf{f} = \rho \frac{\partial^2 \mathbf{u}}{\partial \mathbf{t}^2}$$

Total force is the sum of the forces on *all* the surfaces

Dynamics of 3-D Continuum

3-D Wave Equation



Net force in the x-direction:

$$F_x = \sum_{\text{surfaces}} \left(T_{xx}\, dA_x + T_{xy}\, dA_y + T_{xz}\, dA_z \right)$$

$$\sum_{\text{surface}} T_{xx}\, dA_x = \frac{T_{xx}(x+dx) - T_{xx}(x)}{dx}\, dx\, dy\, dz$$

$$\sum_{\text{surface}} T_{xx}\, dA_x = \frac{\partial T_{xx}}{\partial x}\, dx\, dy\, dz$$

$$F_x = \int\int\int \left[\frac{\partial T_{xx}}{\partial x} + \frac{\partial T_{xy}}{\partial y} + \frac{\partial T_{xz}}{\partial z} \right] dx\, dy\, dz$$

Dynamics of 3-D Continuum

3-D Wave Equation

$$F_x = \int\int\int \left[\frac{\partial T_{xx}}{\partial x} + \frac{\partial T_{xy}}{\partial y} + \frac{\partial T_{xz}}{\partial z} \right] dx\, dy\, dz \qquad T_{ij} = \lambda e\, \delta_{ij} + 2\mu E_{ij}$$

$$F_x = \int_v \rho \frac{\partial^2 \mathbf{u}}{\partial t^2} dx dy dz = \int\int\int \underbrace{\left[(\mu + \lambda) \frac{\partial}{\partial x} (\nabla \cdot \mathbf{u}) + \mu \nabla^2 \mathbf{u_x} \right]}_{\mathbf{f}_x} dx\, dy\, dz$$

Finally, 3-D wave equation….

$$\rho \frac{\partial^2 \mathbf{u}}{\partial t^2}(\mathbf{r}, t) = (\mu + \lambda) \nabla \left[(\nabla \cdot \mathbf{u}(\mathbf{r}, \mathbf{t}) \right] + \mu \nabla^2 \mathbf{u}(\mathbf{r}, \mathbf{t})$$

Dynamics of 3-D Continuum

Fourier Transform of 3-D Wave Equation

$$\rho\frac{\partial^2\mathbf{u}}{\partial t^2}(\mathbf{r},t) = (\mu+\lambda)\nabla\left[(\nabla\cdot\mathbf{u}(\mathbf{r},\mathbf{t})\right] + \mu\nabla^2\mathbf{u}(\mathbf{r},\mathbf{t})$$

Anticipating plane wave solutions, we Fourier Transform the equation….

$$\mathbf{u}(\mathbf{r},\mathbf{t}) = \int\frac{\mathbf{d}\omega}{\mathbf{2}\pi}\int\frac{\mathbf{d^3q}}{(\mathbf{2}\pi)^\mathbf{3}}\mathbf{U}(\mathbf{q},\omega)\mathbf{e}^{\mathbf{i}(\mathbf{q}\cdot\mathbf{r}-\omega\mathbf{t})}$$

$$\rho\omega^2\mathbf{U}(\mathbf{q},\omega) = (\lambda+\mu)\mathbf{q}\left[\mathbf{q}\cdot\mathbf{U}(\mathbf{q},\omega)\right] + \mu\mathbf{q}^\mathbf{2}\mathbf{U}(\mathbf{q},\omega)$$

Three coupled equations for U_x, U_y, and U_z….

Dynamics of 3-D Continuum
Dynamical Matrix

$$\rho\omega^2 \mathbf{U_i}(\mathbf{q},\omega) = (\lambda+\mu)\mathbf{q_i}\,[\mathbf{q}\cdot\mathbf{U}(\mathbf{q},\omega)] + \mu\mathbf{q}^2\mathbf{U_i}(\mathbf{q},\omega)$$

Express the system of equations as a matrix….

$$\rho\omega^2 \begin{bmatrix} \mathbf{U_1} \\ \mathbf{U_2} \\ \mathbf{U_3} \end{bmatrix} = \begin{bmatrix} \mu q^2 + (\lambda+\mu)q_1^2 & (\lambda+\mu)q_1 q_2 & (\lambda+\mu)q_1 q_3 \\ (\lambda+\mu)q_2 q_1 & \mu q^2 + (\lambda+\mu)q_2^2 & (\lambda+\mu)q_2 q_3 \\ (\lambda+\mu)q_3 q_1 & (\lambda+\mu)q_3 q_2 & \mu q^2 + (\lambda+\mu)q_3^2 \end{bmatrix} \begin{bmatrix} \mathbf{U_1} \\ \mathbf{U_2} \\ \mathbf{U_3} \end{bmatrix}$$

Turns the problem into an eigenvalue problem for
the polarizations of the modes (eigenvectors) and
wavevectors **q** (eigenvalues)….

$$\rho\omega^2 \mathbf{U} = \mathbf{D}\,\mathbf{U}$$

Dynamics of 3-D Continuum

Solutions to 3-D Wave Equation

$$\rho\omega^2 \mathbf{U_i}(\mathbf{q},\omega) = (\lambda+\mu)\mathbf{q_i}\,[\mathbf{q}\cdot\mathbf{U}(\mathbf{q},\omega)] + \mu\mathbf{q^2}\mathbf{U_i}(\mathbf{q},\omega)$$

Transverse polarization waves:

$$\mathbf{q}\cdot\mathbf{U}(\mathbf{q},\omega) = \mathbf{0}$$

$$\rho\omega^2 = \mu q^2 \qquad \text{for transverse waves}$$

$$\omega = c_T|\mathbf{q}| \qquad \text{where} \qquad c_T = \sqrt{\frac{\mu}{\rho}}$$

Longitudinal polarization waves:

$$\mathbf{q}\cdot\mathbf{U}(\mathbf{q},\omega) = \mathbf{q}\mathbf{U}$$

$$\rho\omega^2 U = (\lambda+2\mu)q^2 U \qquad \text{for longitudinal waves}$$

$$\omega = c_L|\mathbf{q}| \qquad \text{where} \qquad c_L = \sqrt{\frac{\lambda+2\mu}{\rho}}$$

Dynamics of 3-D Continuum

Summary

1. Dynamical Equation can be solved by inspection

$$\rho\omega^2\mathbf{U}(\mathbf{q},\omega) = (\lambda+\mu)\mathbf{q}\left[\mathbf{q}\cdot\mathbf{U}(\mathbf{q},\omega)\right] + \mu\mathbf{q}^2\mathbf{U}(\mathbf{q},\omega)$$

2. There are 2 transverse and 1 longitudinal polarizations for each q

3. The dispersion relations are linear $\omega = c_i|\mathbf{q}|$

$$c_T = \sqrt{\frac{\mu}{\rho}} \qquad c_L = \sqrt{\frac{\lambda+2\mu}{\rho}}$$

4. The longitudinal sound velocity is always greater than the transverse sound velocity

$$\frac{c_L}{c_T} = \left(\frac{\lambda+2\mu}{\mu}\right)^{1/2} = \left(1+\frac{1}{1-2\nu}\right)^{1/2}$$