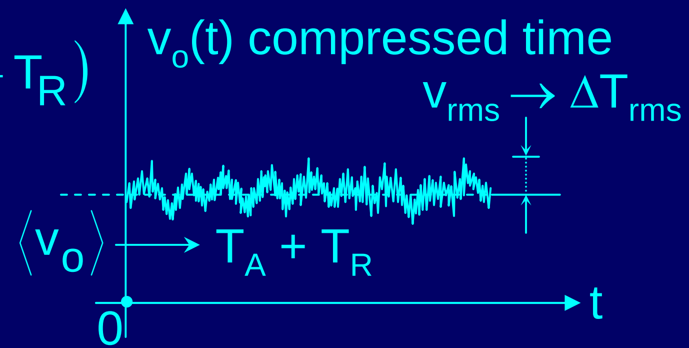
Calculation of receiver sensitivity

$$\Delta T_{rms}(°K) \triangleq \frac{v_{o_{rms}}}{\partial\langle v_o\rangle/\partial T_A}$$

where $\partial\langle v_o\rangle/\partial T_A$ calibrates voltage as temperature

$\langle v_o\rangle \propto (T_A + T_R)$



$$\Phi_o(f)_{DC} \Rightarrow \langle v_o\rangle$$

$$\Phi_o(f)_{AC} \Rightarrow v_{o_{rms}}$$

Approach:

$$v_d(t) \quad \leftrightarrow \quad ?$$

$$\downarrow$$

$$\phi_d(t) \quad \leftrightarrow \quad \Phi_d(f) \Rightarrow \Phi_o(f)$$

Calculation of $\Phi_d(f)$, Power spectrum of $v_i^2(t)$, v_i gaussian

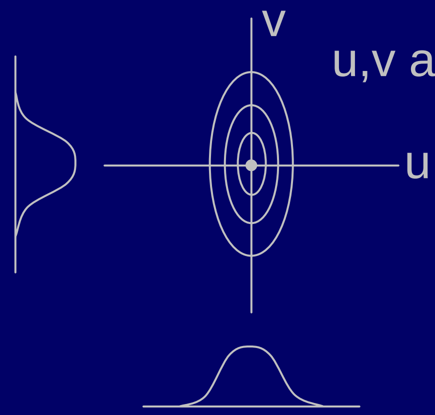
$$\phi_d(\tau) = E[v_d(t) \bullet v_d(t-\tau)] = E[v_i^2(t) v_i^2(t-\tau)]$$

not gaussian

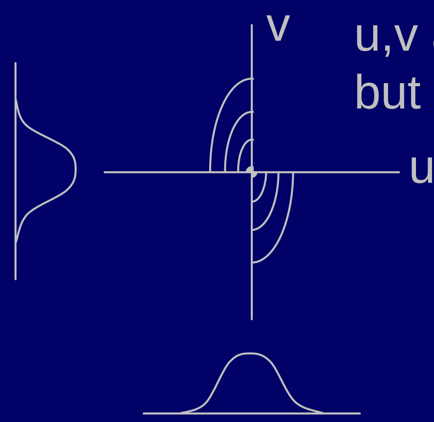
gaussian v_i

It can be shown that:

> E[wxyz] = E[wx]E[yz] + E[wy]E[xz] + E[wz]E[xy]
> if w,x,y,z are jointly gaussian random variables [JGRV]
> with zero mean [JGRVZM]



u,v are JGRVZM



u,v are NOT JGRV,
but are each GRVZM

Calculation of $\Phi_d(f)$, Power spectrum of $v_i^2(t)$, v_i gaussian

$$\phi_d(\tau) = E[v_d(t) \bullet v_d(t-\tau)] = E[v_i^2(t) v_i^2(t-\tau)]$$

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> if w,x,y,z are jointly gaussian random variables [JGRV]
> with zero mean [JGRVZM]

$$\therefore \phi_d(\tau) = \overline{v_i^2(t)}\,\overline{v_i^2(t-\tau)} + 2\overline{v_i(t)v_i(t-\tau)}^2 = \phi_i^2(0) + 2\phi_i^2(\tau) \quad \text{[Ergodic]}$$

$$\updownarrow \qquad\qquad \updownarrow \qquad \updownarrow$$

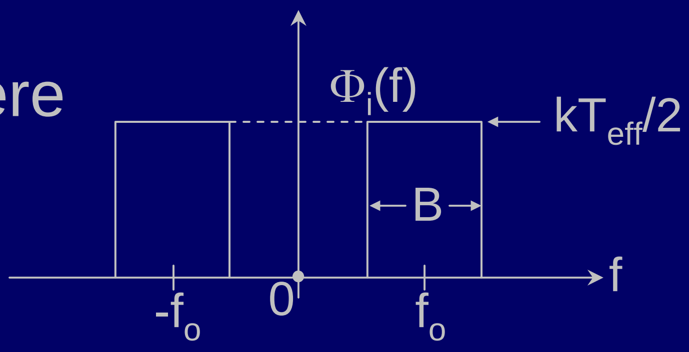
$$\Phi_d(f) = \phi_i^2(0)\delta(f) + 2\Phi_i(f) * \Phi_i^*(f)$$

Evaluation of $\Phi_d(f)$

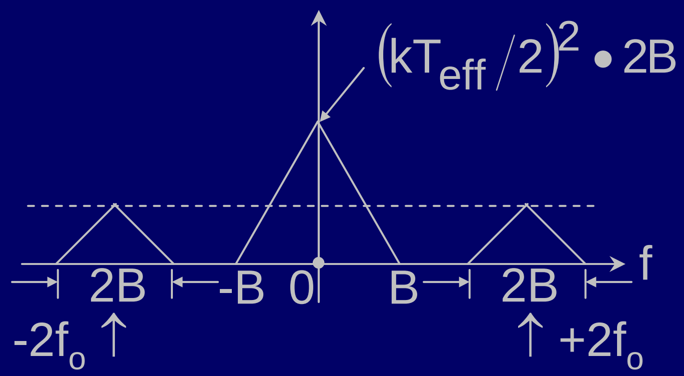
$$\Phi_d(f) = \phi_i^2(0)\delta(f) + 2\Phi_i(f) * \Phi_i^*(f)$$

1) $\phi_i(0) = \overline{v_i^2(t)} = \int_{-\infty}^{\infty} \Phi_i(f)df = kT_{eff}B \quad \left(T_{eff} \triangleq T_A + T_R\right)$

where



2) $\Phi_i(f) * \Phi_i^*(f) =$



Evaluation of $\Phi_d(f)$

$$\Phi_d(f) = \phi_i^2(0)\delta(f) + 2\Phi_i(f) * \Phi_i^*(f)$$

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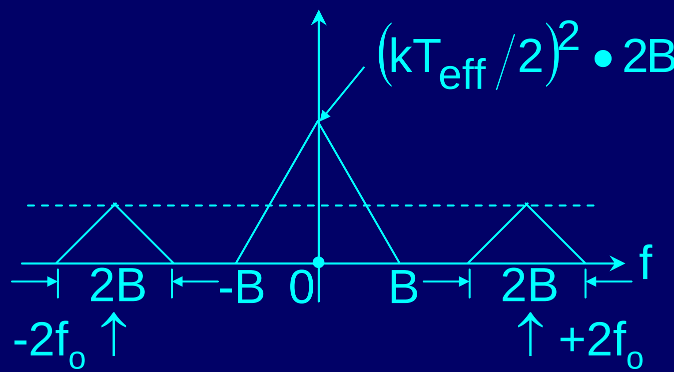
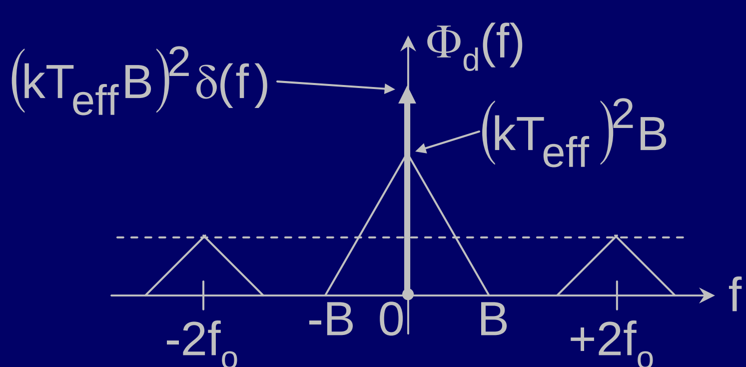
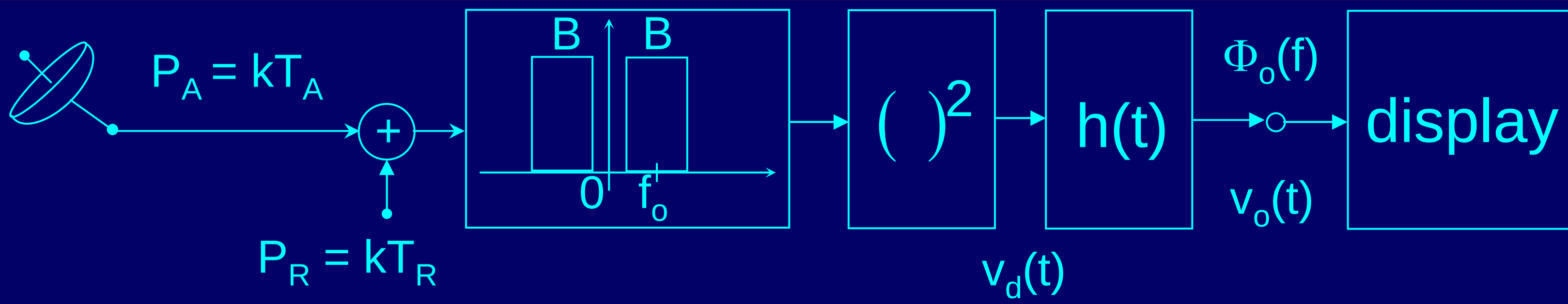


3) Therefore:



Filtered output power density spectrum $\Phi_o(f)$

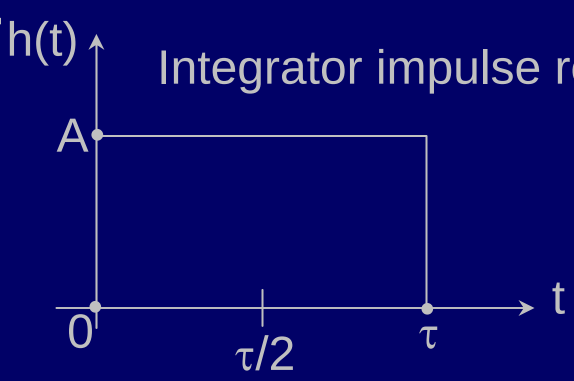


$$v_o(t) = v_d(t) * h(t)$$

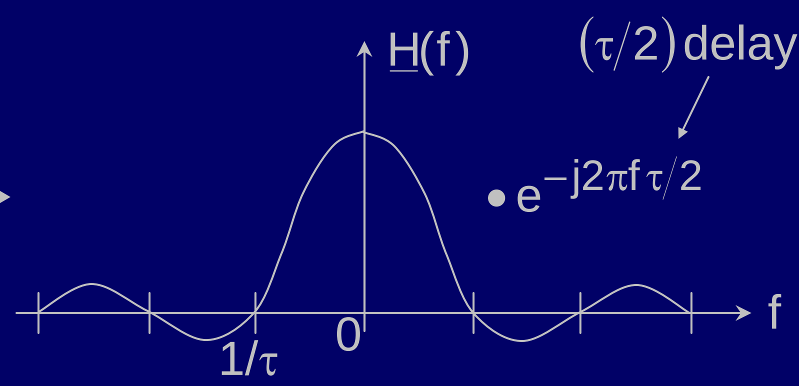
$$\Phi_o(f) = \underbrace{\Phi_d(f)}_{\text{AC+DC terms}} \bullet |H(f)|^2$$

Say:

Integrator impulse response



$\longleftrightarrow$



Filtered output power density spectrum $\Phi_o(f)$

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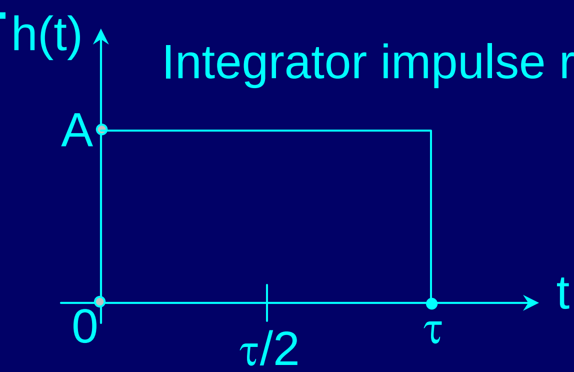
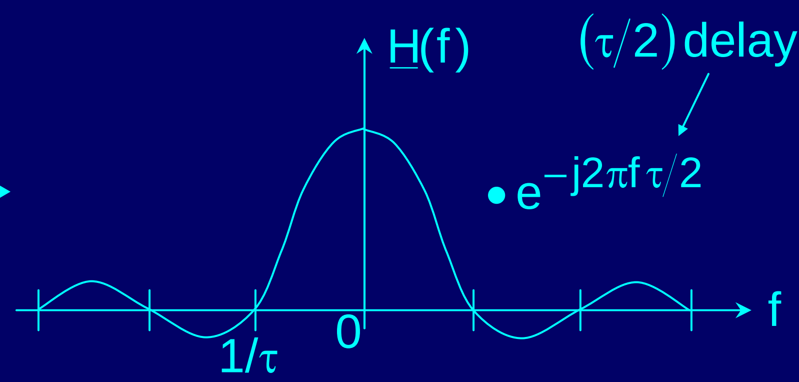
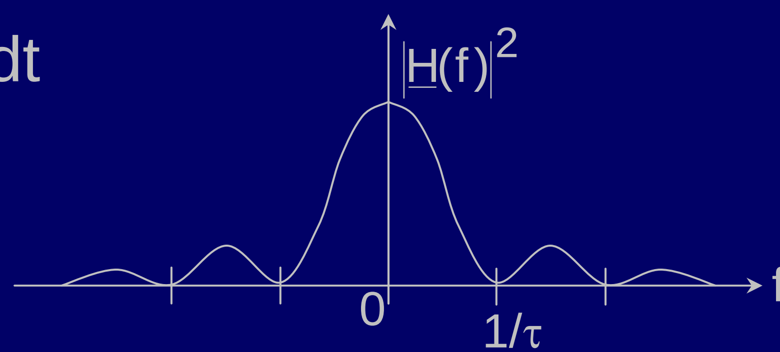
Integrator impulse response





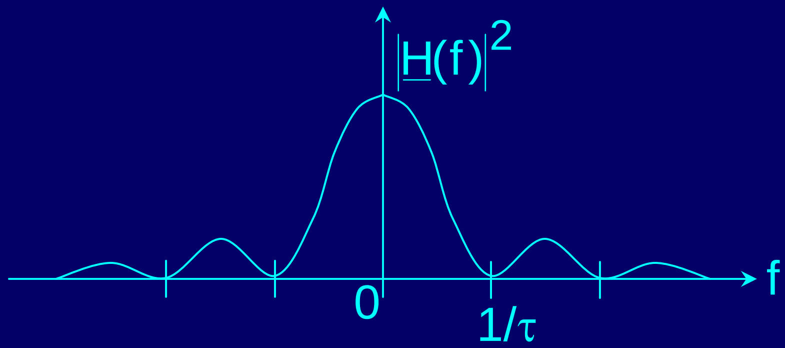
$$\text{Then: } \underline{H}(f=0) = \int_{-\infty}^{\infty} h(t) e^{-j2\pi(f=0)t} dt$$

$$= A\tau$$



(Typically $1/\tau \ll B$)

Filtered output power density spectrum $\Phi_o(f)$



(Typically $1/\tau$ << B)

$$\text{Thus: } \Phi_{o_{DC}}(f) = \left(kT_{eff}B\right)^2 (A\tau)^2 \delta(f) = \text{DC power}$$

$$P_{o_{AC}} = \int_{-\infty}^{\infty} \Phi_{o_{AC}}(f)df \cong \left(kT_{eff}\right)^2 B \bullet \int_{-\infty}^{\infty} |\underline{H}(f)|^2 df$$

Note that if $1/\tau$ << B, only the value of $\Phi_d(f = 0)$ is important, so this integral is trivial.

By Parseval's theorem: $\int_{-\infty}^{\infty} |H(f)|^2 df = \int_{-\infty}^{\infty} h^2(t)dt = A^2\tau$

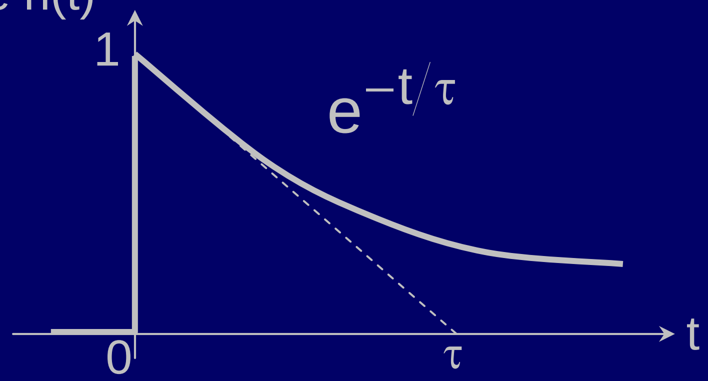
Total-Power Radiometer Sensitivity ΔT_{rms}

$$\Delta T_{rms} = \frac{\sqrt{P_{AC}}}{\left(\partial\sqrt{P_{DC}}/\partial T_A\right)}[^\circ K] = \frac{\sqrt{(kT_{eff})^2 B \bullet A^2 \tau}}{\left(\partial\left[kT_{eff}BA\tau\right]/\partial T_A\right)} = \frac{k\overbrace{T_{eff}}^{T_A + T_R} A\sqrt{B\tau}}{kA\,B\tau}$$

$$\boxed{\therefore \Delta T_{rms} = \frac{T_A + T_R}{\sqrt{B\tau}} \text{ for total-power radiometer}}$$

Effect of different integrator impulse response

Suppose h(t)



Recall $\Phi_o(f) = \Phi_d(f) \bullet |H(f)|^2$

We need to compute

$$H(f=0) = \int_{-\infty}^{\infty} h(t)dt = \tau$$

$$\int_{-\infty}^{\infty} |H(f)|^2 df = \int_{-\infty}^{\infty} h^2(t)dt = \tau/2$$

Then $\Delta T_{rms} = \dfrac{kT_{eff}\sqrt{B\tau/2}}{kB\tau}$

$$\boxed{\Delta T_{rms} = \frac{(T_A + T_R)}{\sqrt{2B\tau}}}$$

Greater sensitivity, but at the expense of a longer memory

Example: Radio telescope receiver

$$\text{Possible}: \; T_A + T_R = 30°K, B = 100\,\text{MHz}$$

$$\text{then}: \; \Delta T_{rms} = 30 \Big/ \sqrt{10^8 \bullet 1\text{sec}} = 0.003°K \Rightarrow 300\;\mu K \text{ for } 100^s$$

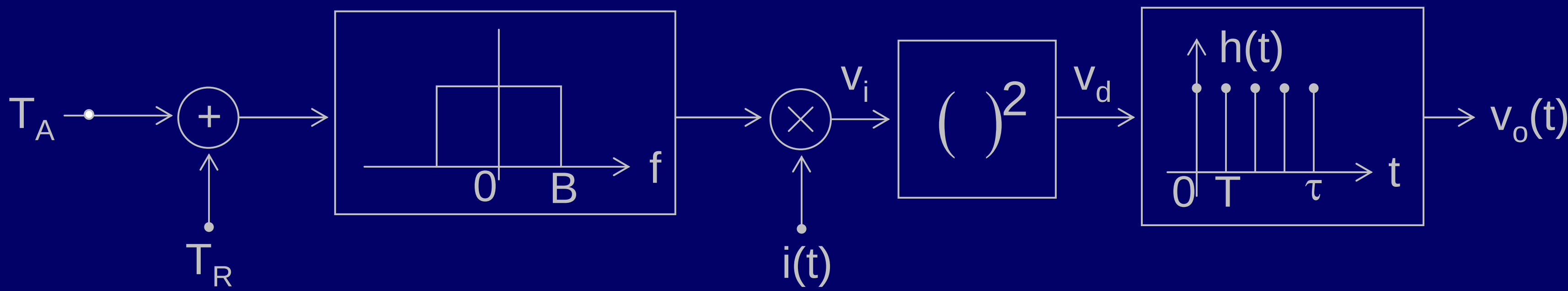
Example: Voice radio, AM

$$\text{If}: \; T_A + T_R = 10{,}000K, B = 10kHz, \tau = 10^{-4}\,\text{sec}$$

$$\text{then}: \; \Delta T_{rms} = 10^4 \Big/ \sqrt{10^4 \bullet 10^{-4}} = 10^4 K = T_A + T_R$$

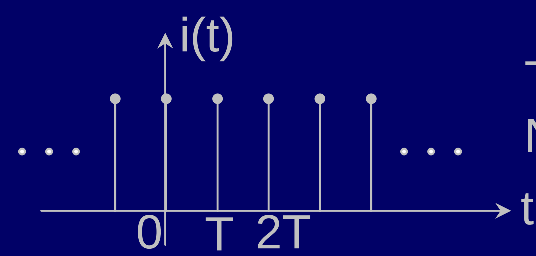
Receiver sensitivity derivation: sampled signals

Sampling-Theorem approach for the total-power radiometer



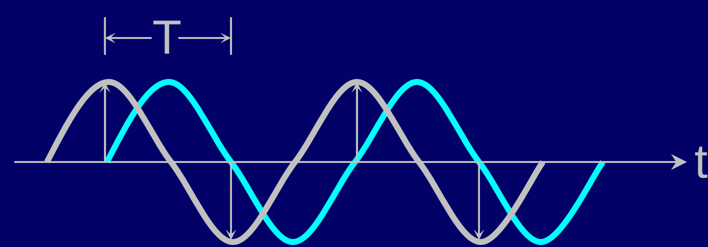
$T < 1/2B \Rightarrow$ pulse correlation

$T > 1/2B \Rightarrow$ lost information



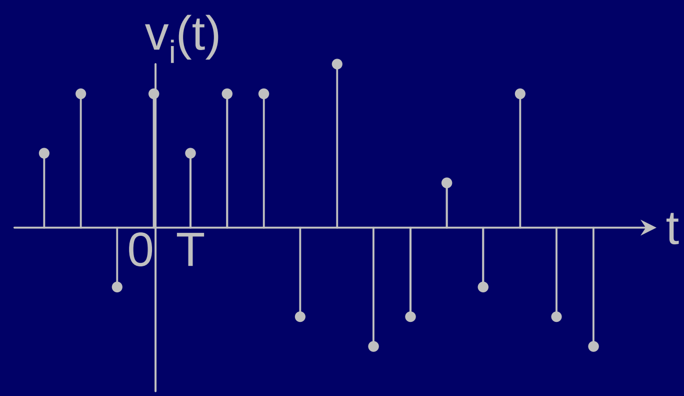
T = 1/2B is the Nyquist rate

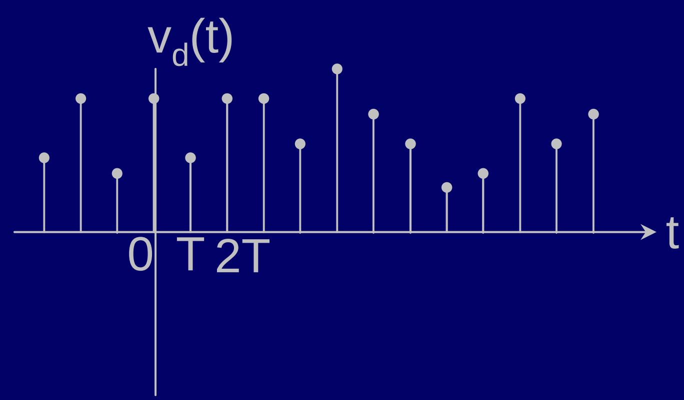
Nyquist sampling: e.g.



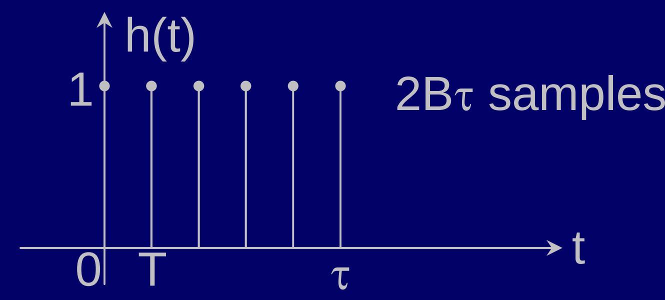
Highest f (=B here) T = 1/2B

Computation of ΔT_{rms} for a sampled system





Boxcar having $\tau/T = 2B\tau$ samples



$$\sum_{\text{boxcar}}^{2B\tau} v_d \Rightarrow \text{Gaussian if } 2B\tau = \tau/T >> 1$$

(central limit theorem)

$$\Delta T_{rms} = \frac{v_{o_{rms_{AC}}}}{\left[\frac{\partial\langle v_o\rangle}{\partial T_A}\right]} = \text{(output fluctuation/scale calibration)}$$

$$\text{Variance of } v_o = \underbrace{2B\tau}_{\#\text{ samples}} \bullet \underbrace{\sigma_d^2}_{\text{variance of } v_d}$$

$$\Delta T_{rms} = \frac{v_{o_{rms_{AC}}}}{\left[\frac{\partial\langle v_o\rangle}{\partial T_A}\right]} = \text{(output fluctuation/scale calibration)}$$

$$\text{Variance of } v_o = \underbrace{2B\tau}_{\#\text{ samples}} \bullet \underbrace{\sigma_d^2}_{\text{variance of } v_d}$$

$$\sigma_d^2 \triangleq \overline{\left(v_d - \overline{v}_d\right)^2} = \overline{\left(v_i^2 - \overline{v_i^2}\right)^2} \qquad \text{(where } v_i = \text{JGRVZM)}$$

$$= \overline{v_i^4} - 2\left(\overline{v_i^2}\right)^2 + \left(\overline{v_i^2}\right)^2 = \overline{v_i^4} - \left(\overline{v_i^2}\right)^2$$

$$\text{Recall: } \overline{x^n} = 1 \bullet 3 \bullet 5 \bullet \bullet (n-1), \text{ if even; } \overline{x^n} = 0, \text{ if n odd}$$

(where x = JGRVZM)

$$\text{Let: } \overline{x^2} \equiv 1 \text{ here and } \overline{v_i^2} = T_{eff} \bullet a \bullet \overline{x^2} \text{ (this equation defines "a")}$$

$$\text{Thus: } \sigma_d^2 = \overline{v_i^4} - \left(\overline{v_i^2}\right)^2 = T_{eff}^2 a^2 \left[\underbrace{\overline{x^4}}_{3} - \left(\underbrace{\overline{x^2}}_{1}\right)^2\right] = 2T_{eff}^2 a^2$$

and the variance of $v_o = 2B\tau \bullet 2T_{eff}^2 a^2$

Computation of ΔT_{rms} for a sampled system

$$\text{variance of } v_o = 2B\tau \bullet 2T_{eff}^2 a^2$$

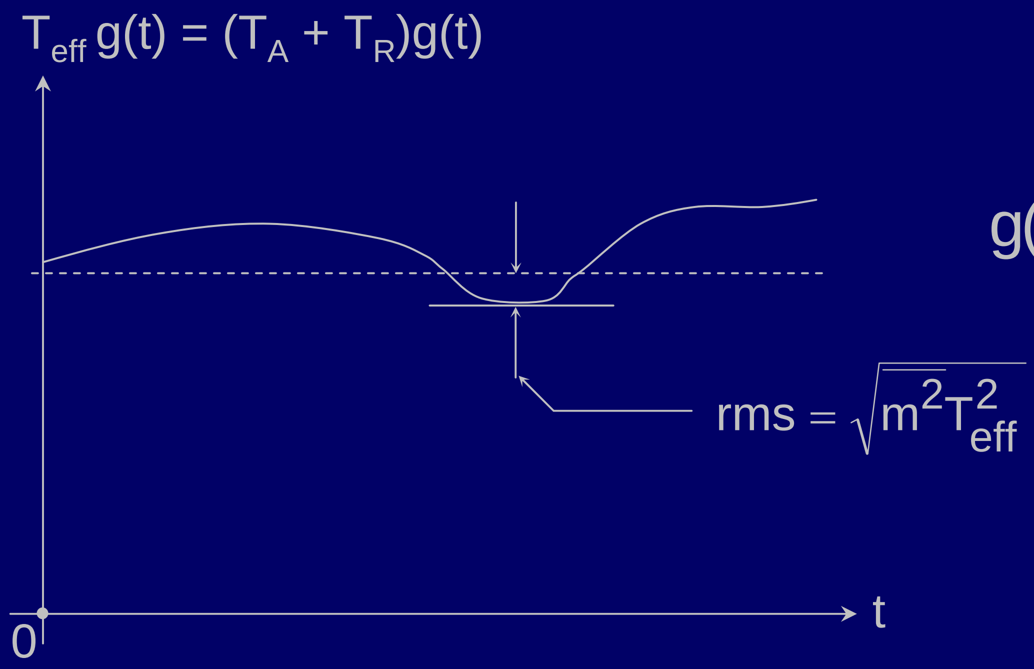
$$\bar{v}_o = 2B\tau \bullet \overline{v_i^2} = 2B\tau \bullet T_{eff}\, a$$

$$\therefore \Delta T_{rms} = \frac{\sqrt{\text{variance of } v_o}}{\partial \bar{v}_o / \partial T_A} = \frac{T_{eff}\, a \sqrt{4B\tau}}{2B\tau a}$$

$$\boxed{\Delta T_{rms} = T_{eff} / \sqrt{B\tau}}$$ as before

Note: # samples = $2B\tau$, $\sqrt{\text{variance}} \propto 1/\sqrt{B\tau}$

Gain fluctuations in total-power radiometers

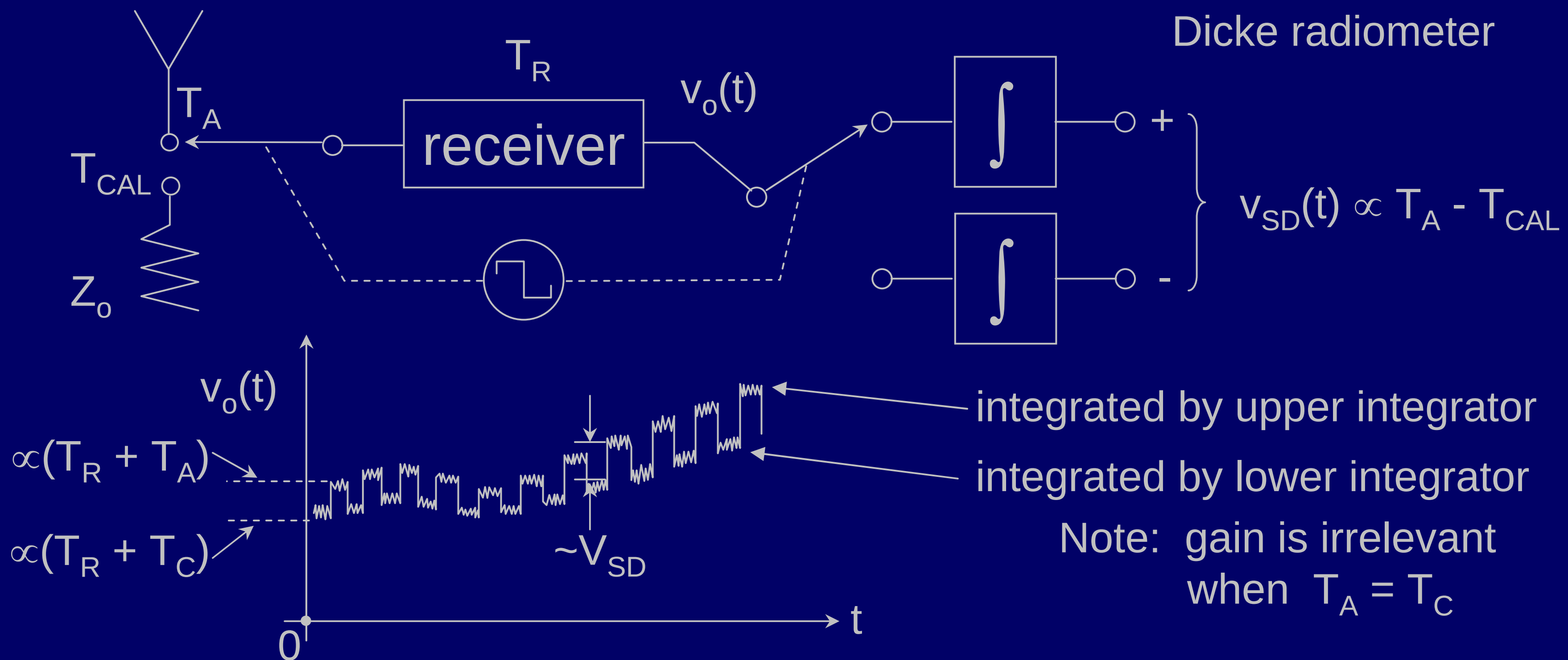


$$g(t) \cong 1 + m(t),\ |m| << 1$$

$$\Delta T_{rms} \cong \sqrt{(\Delta T_{thermal})^2 + \overline{m^2} T_{eff}^2} = T_{eff} \sqrt{\frac{1}{B\tau} + \overline{m^2}}$$

(Note: 0.1% gain fluct. @ T_{eff} = 2000K $\Rightarrow$ 2K!)

One solution to gain variations: "Synchronous detection"

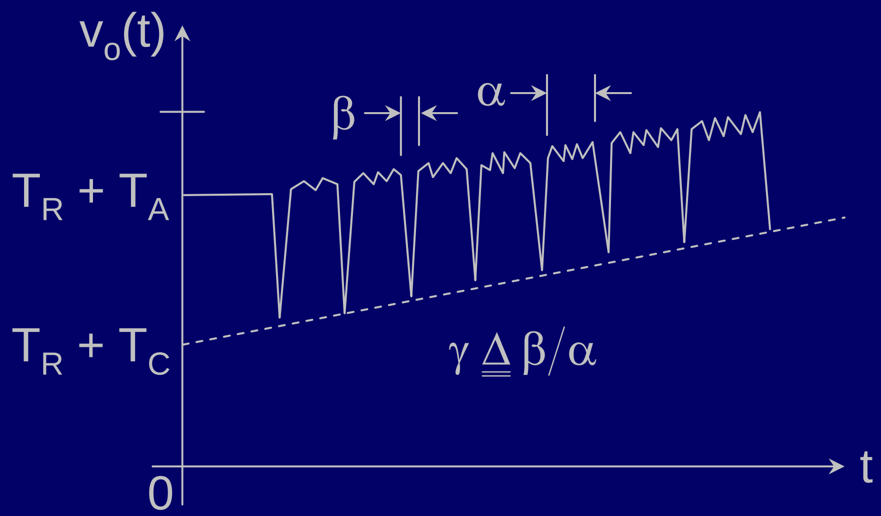


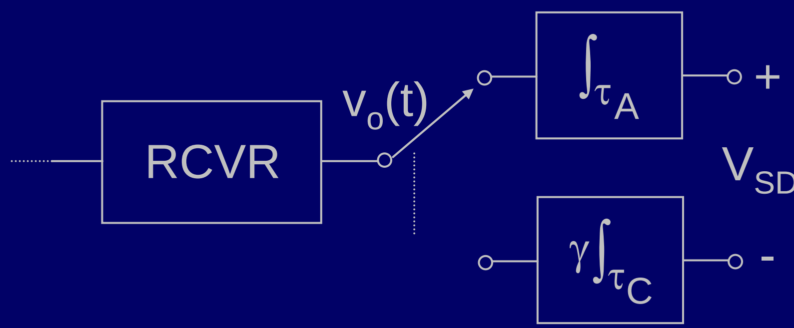
$$v_{SD_{rms}} = \text{unchanged (looking at same signal all the time)}$$

$$\text{but } \partial v_{SD} / \partial T_{eff} = \frac{1}{2} \text{ former value (we view } T_A \text{ half the time)}$$

$$\boxed{\therefore \Delta T_{rms_{Dicke}} = 2T_{eff} / \sqrt{B\tau} \text{ (at null only)}}$$

Asymmetric Dicke radiometer



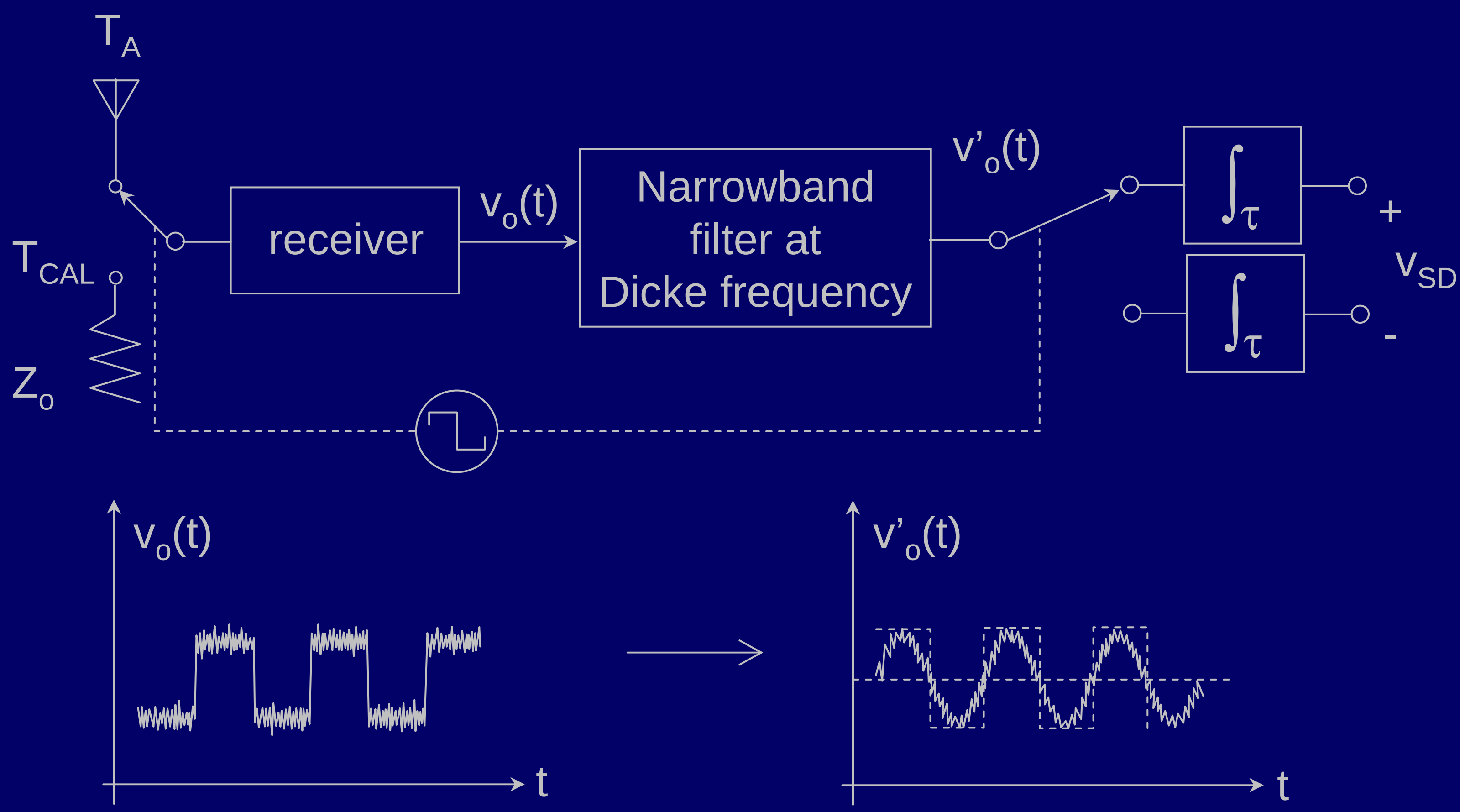


Want: τ_A << desired-signal fluctuation time constant

$\tau_c > \tau_A$ (typically $\tau_c \geq \gamma\tau_A$)

Integration times τ_A and τ_c should be shorter than the fluctuation times of the desired signal and system gain, respectively.

Filtered Dicke radiometer

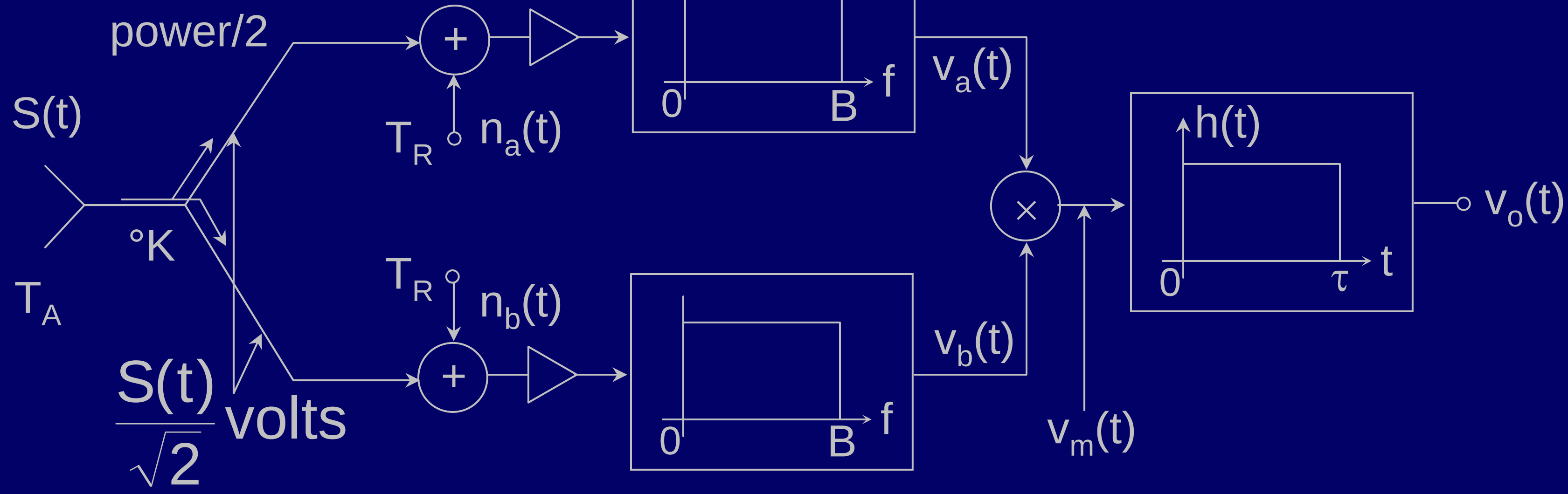


$$\Delta T_{rms} = \frac{\pi}{\sqrt{2}} T_{eff} \Big/ \sqrt{B\tau}$$

narrowband Dicke $\quad \pi/\sqrt{2} = 2.22$

Correlation radiometer

Typical circuit:



Uses:

1) To reduce radiometric gain modulation effects (similar to Dicke receiver)
2) As a correlator

Correlator power density spectrum, pre-integrator:

$$\phi_m(\tau) = E[v_a(t)v_b(t)v_a(t-\tau)v_b(t-\tau)]$$

Correlator power density spectrum, pre-integrator

$$\phi_m(\tau) = E\left[v_a(t)v_b(t)v_a(t-\tau)v_b(t-\tau)\right]$$

$$= E\left[\left(\frac{S_1}{\sqrt{2}} + n_{a_1}\right)\left(\frac{S_1}{\sqrt{2}} + n_{b_1}\right)\left(\frac{S_2}{\sqrt{2}} + n_{a_2}\right)\left(\frac{S_2}{\sqrt{2}} + n_{b_2}\right)\right]$$

$$\text{Where } S_1 \triangleq S(t),\ n_1 \triangleq n(t),\ S_2 \triangleq S(t-\tau),\ n_2 \triangleq n(t-\tau)$$

$$\text{All JGRVZM, so: } \overline{ABCD} = \overline{AB}\bullet\overline{CD} + \overline{AC}\bullet\overline{BD} + \overline{AD}\bullet\overline{BC}$$

$$\phi_m(\tau) = \underbrace{\frac{1}{4}\phi_s^2(0) + \frac{1}{2}\phi_s^2(\tau)}_{S\times S\text{ terms}} + \underbrace{\phi_s(\tau)\phi_n(\tau)}_{S\times n} + \underbrace{\phi_n^2(\tau)}_{n\times n}$$

$$\Phi_m(f) = \underbrace{\frac{1}{4}\phi_s^2(0)\delta(f) + \frac{1}{2}\Phi_s(f)*\Phi_s(f)}_{S\times S} + \underbrace{\Phi_s(f)*\Phi_n(f)}_{S\times n} + \underbrace{\Phi_n(f)*\Phi_n(f)}_{n\times n}$$

Sensitivity of correlation radiometer

$$\Phi_m(f) = \underbrace{\frac{1}{4}\phi_s^2(0)\delta(f) + \frac{1}{2}\Phi_s(f) * \Phi_s(f)}_{S\times S} + \underbrace{\Phi_s(f) * \Phi_n(f)}_{S\times n} + \underbrace{\Phi_n(f) * \Phi_n(f)}_{n\times n}$$

P_{dc} follows from $\frac{1}{4}\phi_s^2(0)\delta(f)$, and P_{ac} from the other terms

$$\Delta T_{rms} = \frac{\sqrt{P_{ac}}}{\partial\sqrt{P_{dc}}\,/\,\partial T_A} = \frac{T_{eff}}{\sqrt{B\tau}}$$

$$\text{where } T_{eff}^2 = T_A^2 + 2T_A T_R + 2T_R^2$$

$\Delta T_{rms} \cong \dfrac{\sqrt{2}T_R}{\sqrt{B\tau}}$ for the weak-signal case ($T_A \ll T_R$)

$\Delta T_{rms} \cong T_A/\sqrt{B\tau}$ for the strong-signal case ($T_A \gg T_R$)

Summary – radiometer sensitivity

Radiometer type	ΔT_{rms}	T_{eff}^2	Relative sensitivity to small fluctuations
Total power	$T_{eff} / \sqrt{B\tau}$	$(T_A + T_R)^2$	1
Correlation	$T_{eff} / \sqrt{B\tau}$	$(T_A + T_R)^2 + T_R^2$	$\sqrt{2}$
Dicke	$2T_{eff} / \sqrt{B\tau}$	$(T_A + T_R)^2$ at null	2 at null
Dicke narrowband post detector	$\frac{\pi}{\sqrt{2}} T_{eff} / \sqrt{B\tau}$	$(T_A + T_R)^2$	2.22