6.641 Electromagnetic Fields, Forces, and Motion
Spring 2009

Lecture 14: Fields and Moving Media

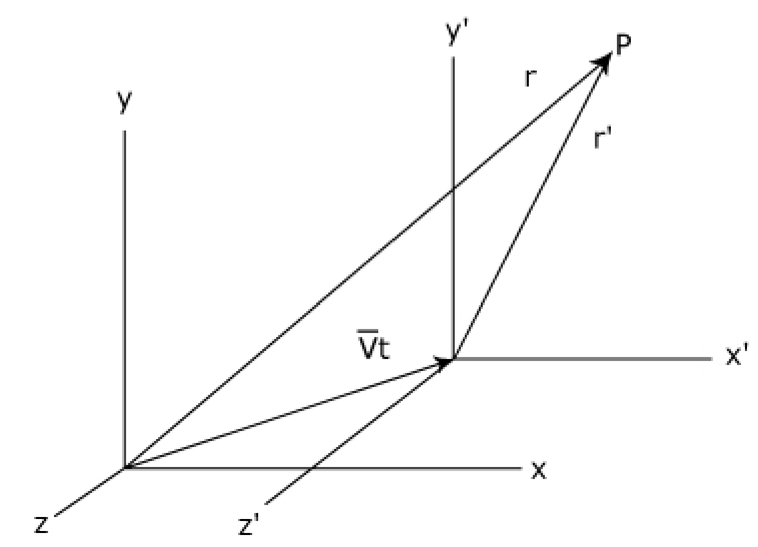
I. Galilean Time and Space Transformations

$t=t', \; \bar{r}' = \bar{r} - \bar{v}\,t$

$x' = x - v_x\,t$

$y' = y - v_y\,t$

$z' = z - v_z\,t$



Two inertial coordinate systems in relative motion.

$$\nabla = \bar{i}_x \frac{\partial}{\partial x} + \bar{i}_y \frac{\partial}{\partial y} + \bar{i}_z \frac{\partial}{\partial z}$$

$$\nabla' = \bar{i}_x \frac{\partial}{\partial x'} + \bar{i}_y \frac{\partial}{\partial y'} + \bar{i}_z \frac{\partial}{\partial z'}$$

$$\nabla' f' = \bar{i}_x \frac{\partial f'}{\partial x'} + \bar{i}_y \frac{\partial f'}{\partial y'} + \bar{i}_z \frac{\partial f'}{\partial z'} \qquad \left(f'\left(x', y', z', t'\right)\right)$$

$$\left.\frac{\partial f'}{\partial x}\right|_{y,z,t} = \frac{\partial f'}{\partial x'}\underbrace{\frac{\partial x'}{\partial x}}_{1} + \frac{\partial f'}{\partial y'}\overbrace{\frac{\partial y'}{\partial x}}^{0} + \frac{\partial f'}{\partial z'}\overbrace{\frac{\partial z'}{\partial x}}^{0} + \frac{\partial f'}{\partial t'}\overbrace{\frac{\partial t'}{\partial x}}^{0} \quad \left(f'(x,y,z,t)\right)$$

$$\left.\frac{\partial f'}{\partial x}\right|_{y,z,t} = \left.\frac{\partial f'}{\partial x'}\right|_{y',z',t'} \quad , \quad \left.\frac{\partial f'}{\partial y}\right|_{x,z,t} = \left.\frac{\partial f'}{\partial y'}\right|_{x',z',t'} \quad , \quad \left.\frac{\partial f'}{\partial z}\right|_{x,y,t} = \left.\frac{\partial f'}{\partial z'}\right|_{x',y',t'}$$

$$\nabla' f' = \nabla f'$$

$$\nabla' \bullet \overline{A}' = \nabla \bullet \overline{A}'$$

$$\nabla' \text{x}\, \overline{A}' = \nabla \times \overline{A}'$$

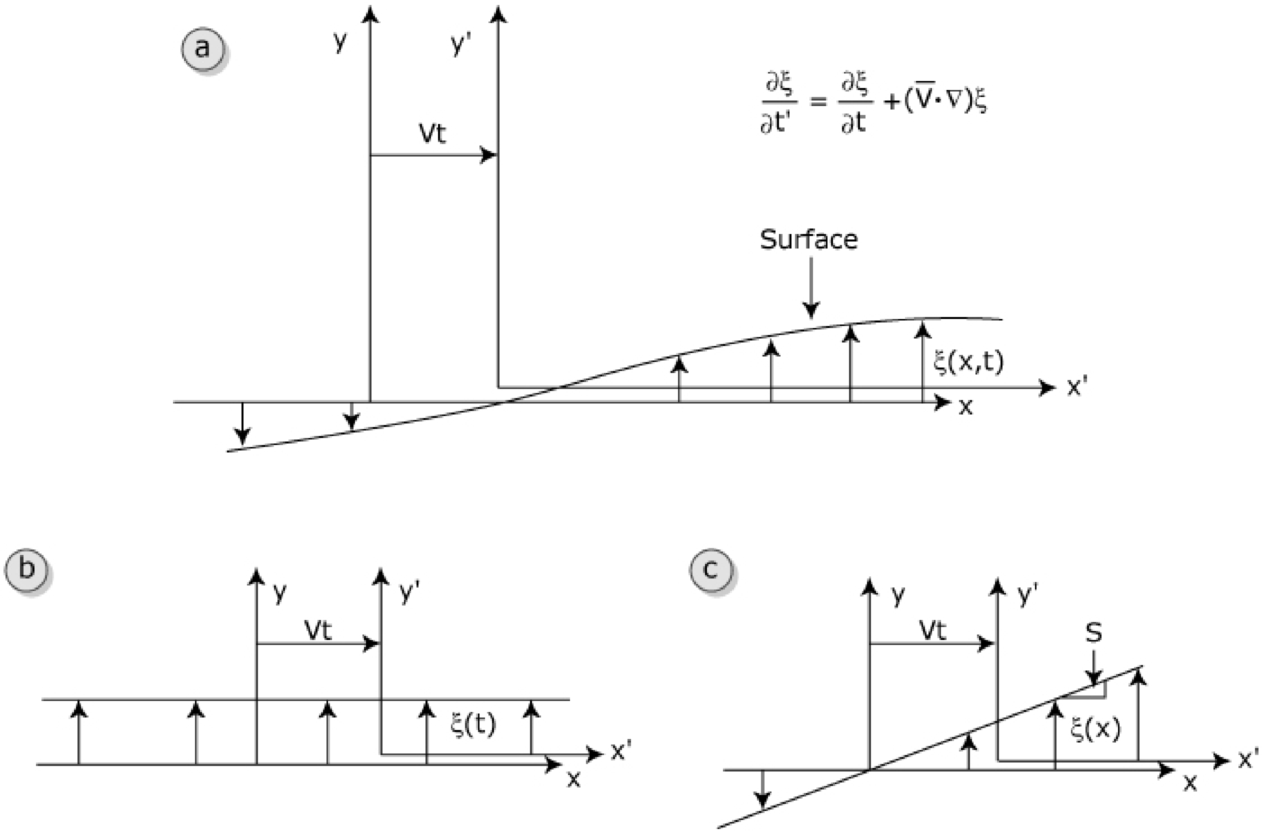
$$\left.\frac{\partial f'}{\partial t}\right|_{x,y,z} = \frac{\partial f'}{\partial t'}\underbrace{\frac{\partial t'}{\partial t}}_{1} + \frac{\partial f'}{\partial x'}\underbrace{\frac{\partial x'}{\partial t}}_{-v_x} + \frac{\partial f}{\partial y'}\underbrace{\frac{\partial y'}{\partial t}}_{-v_y} + \frac{\partial f'}{\partial z'}\underbrace{\frac{\partial z'}{\partial t}}_{-v_z}$$

$$\left.\frac{\partial f'}{\partial t}\right|_{x,y,z} = \frac{\partial f'}{\partial t'} - v_x \frac{\partial f'}{\partial x'} - v_y \frac{\partial f'}{\partial y'} - v_z \frac{\partial f'}{\partial z'}$$

$$= \frac{\partial f'}{\partial t'} - \left(\overline{v} \bullet \nabla'\right) f' = \frac{\partial f'}{\partial t'} - \left(\overline{v} \bullet \nabla\right) f'$$

$$\frac{\partial f'}{\partial t'} = \frac{\partial f'}{\partial t} + \left(\overline{v} \bullet \nabla\right) f'$$

$$\frac{\partial \overline{A}'}{\partial t'} = \frac{\partial \overline{A}'}{\partial t} + \left(\overline{v} \bullet \nabla\right) \overline{A}'$$



(a) A surface described by $y=\xi(x,t)$ has an elevation above the x-z plane which is the same whether viewed from the moving (primed) frame or the fixed frame ($\xi'=\xi$); (b) ξ is independent of position so that only the first term makes a contribution to $\partial\xi/\partial t'$; (c) ξ is independent of time and only the second term makes a contribution to $\partial\xi/\partial t'$.

II. Transformations for MQS Systems

$$\nabla \times \bar{H} = \bar{J}_f \qquad \nabla' \times \bar{H}' = \bar{J}_f'$$

$$\nabla \cdot \bar{B} = 0 \qquad \nabla' \cdot \bar{B}' = 0$$

$$\nabla \cdot \bar{J}_f = 0 \qquad \nabla' \cdot \bar{J}_f' = 0$$

$$\nabla \times \bar{E} = -\frac{\partial \bar{B}}{\partial t} \qquad \nabla' \times \bar{E}' = -\frac{\partial \bar{B}'}{\partial t'} = -\frac{\partial \bar{B}'}{\partial t} - \left(\bar{v} \cdot \nabla\right)\bar{B}'$$

$$\bar{B} = \mu_0\left(\bar{H} + \bar{M}\right) \qquad \bar{B}' = \mu_0\left(\bar{H}' + \bar{M}'\right)$$

Useful vector identity:

$$\nabla \times \left(\bar{a} \times \bar{b}\right) = \left(\bar{b} \cdot \nabla\right)\bar{a} - \left(\bar{a} \cdot \nabla\right)\bar{b} + \bar{a}\left(\nabla \cdot \bar{b}\right) - \bar{b}\left(\nabla \cdot \bar{a}\right)$$

Take $\bar{a} = \bar{v}$, (Constant Vector), $\bar{b} = \bar{B}'$

$$\nabla \times \left(\bar{v} \times \bar{B}'\right) = \underbrace{\left(\bar{B}' \cdot \nabla\right)\bar{v}}_{\bar{v}\text{ is constant}}^{\;0} - \left(\bar{v} \cdot \nabla\right)\bar{B}' + \bar{v}\underbrace{\left(\nabla \cdot \bar{B}'\right)}_{\text{Gauss' Law}}^{\;0} - \bar{B}'\underbrace{\left(\nabla \cdot \bar{v}\right)}_{\bar{v}\text{ is constant}}^{\;0}$$

$$\nabla \times \left(\bar{v} \times \bar{B}'\right) = -\left(\bar{v} \cdot \nabla\right)\bar{B}'$$

$$\nabla \times \bar{E}' = -\frac{\partial \bar{B}'}{\partial t} + \nabla \times \left(\bar{v} \times \bar{B}'\right) \Rightarrow \nabla \times \underbrace{\left(\bar{E}' - \bar{v} \times \bar{B}'\right)}_{\bar{E}} = -\frac{\partial \bar{B}'}{\partial t}$$

$$\bar{H}' = \bar{H}$$

$$\bar{B}' = \bar{B}$$

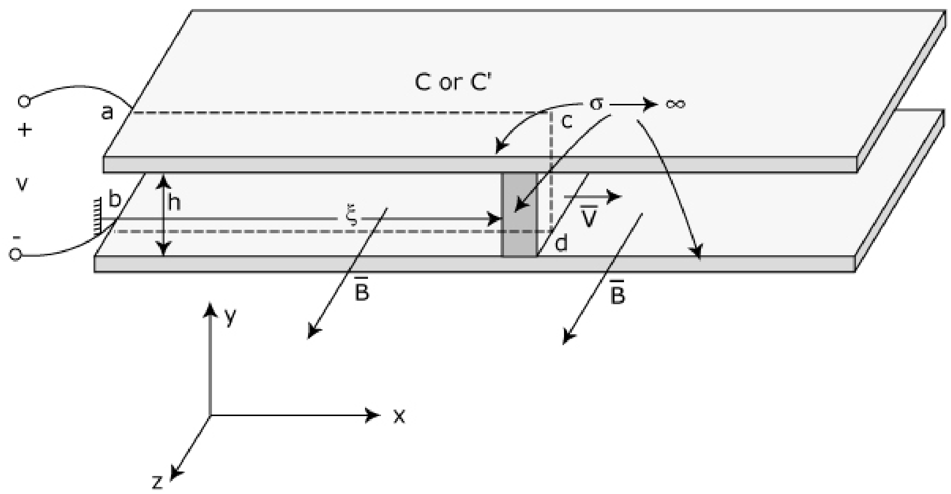
$$\bar{J}_f' = \bar{J}_f$$

$$\bar{E}' = \bar{E} + \bar{v} \times \bar{B}$$

$$\bar{M}' = \bar{M}$$

Note: $\bar{f} = q\left(\bar{E} + \bar{v} \times \bar{B}\right) = q\bar{E}' = \bar{f}'$

III. Moving Media MQS Problem



A pair of parallel perfectly conducting plates are short-circuited by a moving perfectly conducting bar. Because of the magnetic field B, a voltage v is induced which can be computed either by integrating the induction equation around the fixed loop C' that passes through the bar or by integrating the induction equation around a loop C that expands in area as the bar moves to the right. The electric field transformation $\bar{E}' = \bar{E} + \bar{V} \times \bar{B}$ guarantees that both integrations will give the same result.

Moving Contour C

$$\oint_C \bar{E}' \cdot \overline{dl} = -\frac{d}{dt}\int_S \bar{B} \cdot \bar{n}\, da$$

$$-v = -\frac{d}{dt}\left[-B h \xi\right]$$

$\bar{B}$ and $\bar{n}$ in opposite directions

$$v = -B h \frac{d\xi}{dt} = -B h V$$

Stationary Contour C'

$$\oint_{C'} \bar{E} \cdot \overline{dl} = -\frac{d}{dt}\int_{S'} \bar{B} \cdot \bar{n}\, da = 0$$

$\bar{E}' = 0 = \bar{E} + \bar{V} \times \bar{B}$ in moving perfect conductor

$$-v + \left(\bar{V} \times \bar{B}\right)_y h = 0$$

$$v = -B h V$$

IV. Transformations for EQS Systems

$$\nabla \times \bar{E} = 0 \qquad \nabla' \times \bar{E}' = 0$$

$$\nabla \cdot \bar{D} = \rho_f \qquad \nabla' \cdot \bar{D}' = \rho_f'$$

$$\nabla \cdot \bar{J}_f = -\frac{\partial \rho_f}{\partial t} \qquad \nabla' \cdot \bar{J}'_f = -\frac{\partial \rho_f'}{\partial t'}$$

$$\nabla \times \bar{H} = \bar{J}_f + \frac{\partial \bar{D}}{\partial t} \qquad \nabla' \times \bar{H}' = \bar{J}_f' + \frac{\partial D'}{\partial t'}$$

$$\bar{D} = \varepsilon_0 \bar{E} + \bar{P} \qquad \bar{D}' = \varepsilon_0 \bar{E}' + \bar{P}'$$

$$\nabla \times \bar{E}' = 0$$

$$\nabla \cdot \bar{D}' = \rho_f'$$

$$\nabla \cdot \bar{J}_f' = -\frac{\partial \rho_f'}{\partial t'} - \left(\bar{v} \cdot \nabla\right) \rho_f'$$

$$\nabla \times \bar{H}' = \bar{J}_f' + \frac{\partial}{\partial t} \bar{D}' + \left(\bar{v} \cdot \nabla\right) \bar{D}'$$

$$\nabla \times \left(\bar{v} \times \bar{D}'\right) = \underbrace{\left(\bar{D}' \cdot \nabla\right)\bar{v}}_{0} - \left(\bar{v} \cdot \nabla\right)\bar{D}' + \bar{v}\underbrace{\left(\nabla \cdot \bar{D}'\right)}_{\rho_f'} - \bar{D}'\underbrace{\left(\nabla \cdot \bar{v}\right)}_{0}$$

$$\left(\bar{v} \cdot \nabla\right)\bar{D}' = \rho_f' \, \bar{v} - \nabla \times \left(\bar{v} \times \bar{D}'\right)$$

$$\nabla \times \bar{H}' = \bar{J}_f' + \frac{\partial \bar{D}'}{\partial t} + \rho_f' \, \bar{v} - \nabla \times \left(\bar{v} \times \bar{D}'\right)$$

$$\nabla \times \underbrace{\left(\bar{H}' + \bar{v} \times \bar{D}'\right)}_{\bar{H}} = \underbrace{\bar{J}_f' + \rho_f' \, \bar{v}}_{\bar{J}_f} + \frac{\partial \bar{D}'}{\partial t}$$

$$\bar{E}' = \bar{E}$$

$$\bar{D}' = \bar{D}$$

$$\rho_f' = \rho_f$$

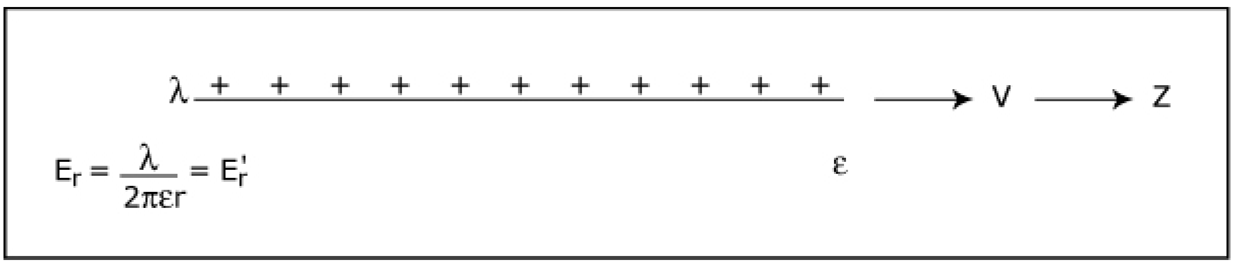
$$\overline{H}' = \overline{H} - \overline{v} \times \overline{D}'$$

$$\overline{J}_f' = \overline{J}_f - \rho_f \overline{v}$$

(Note: $\nabla \bullet \overline{J}_f' = \nabla \bullet \overline{J}_f - \nabla \bullet \left(\rho_f \overline{v}\right)$
$= \nabla \bullet \overline{J}_f - \left(\overline{v} \bullet \nabla\right)\rho_f - \rho_f \cancel{\nabla \bullet \overline{v}}^{\,0}$)

$$\overline{P}' = \overline{P}$$

V. Moving Line Charge Representation Problem



In moving frame: $\overline{H}' = 0 \Rightarrow \overline{H} - \overline{v} \times \overline{D} = 0$

$$\overline{H} = \overline{v} \times \overline{D} = V\,\overline{i}_z \times \left(\frac{\lambda}{2\pi r}\,\overline{i}_r\right) = \frac{V\lambda}{2\pi r}\,\overline{i}_\phi$$

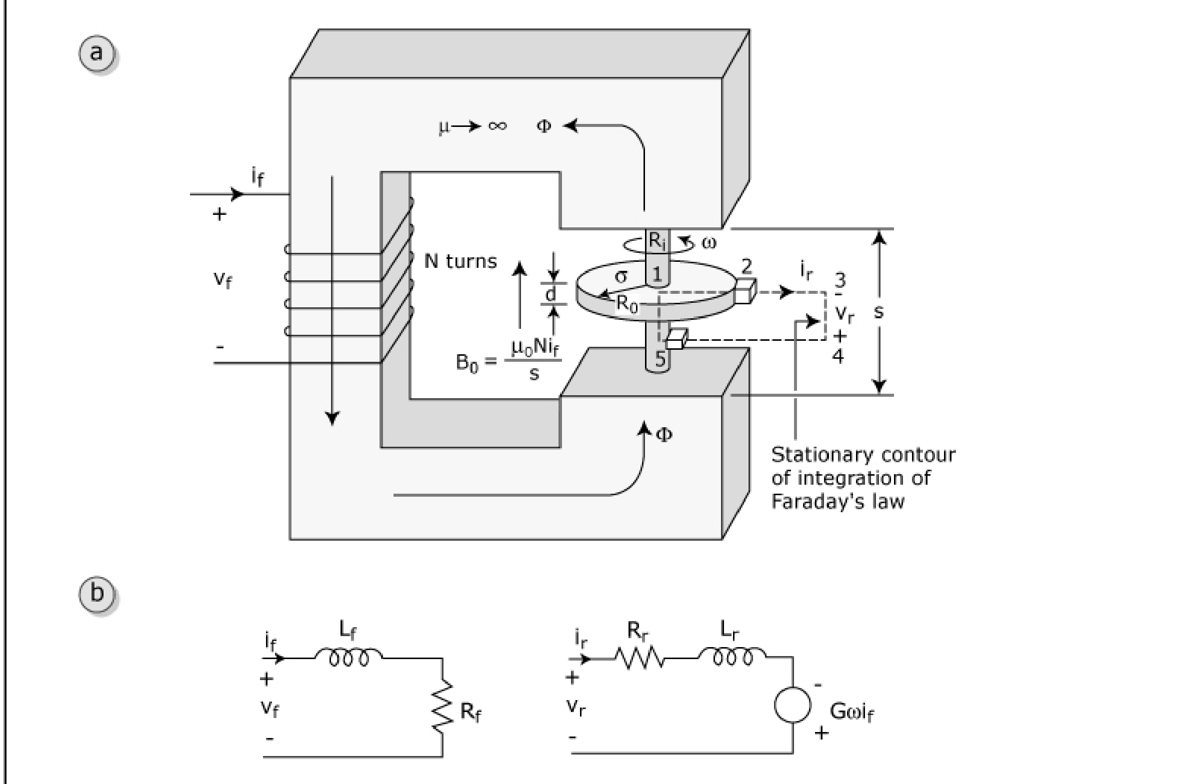
$$\overline{J}' = 0 = \overline{J}_f - \rho_f \overline{v}$$

$$\lambda V = \lambda V \frac{\text{coul}}{\cancel{m}} \frac{\cancel{m}}{s} = \lambda V \text{ amperes} = I = \lambda V$$

$$H_\phi = \frac{I}{2\pi r} = \frac{\lambda V}{2\pi r}$$

	Differential Equations	Transformations	Boundary Conditions
Magnetic Field System	$\nabla \times H = J_f$ $\nabla \cdot B = 0$ $\nabla \cdot J_f = 0$ $\nabla \times E = -\frac{\partial B}{\partial t}$ $B = \mu_0(H + M)$	$H' = H$ $B' = B$ $J_f' = J_f$ $E' = E + v \times B$ $M' = M$	$n \times (H^a - H^b) = K_f$ $n \cdot (B^a - B^b) = 0$ $n \cdot (J_f^a - J_f^b) + \nabla_\Sigma \cdot K_f = 0$ $n \times (E^a - E^b) = v_n (B^a - B^b)$
Electric Field System	$\nabla \times E = 0$ $\nabla \times D = \rho_f$ $\nabla \cdot J_f = -\frac{\partial \rho_f}{\partial t}$ $\nabla \times H = J_f + \frac{\partial D}{\partial t}$ $D = \varepsilon_0 E + P$	$E' = E$ $D' = D$ $\rho'_f = \rho_f$ $J'_f = J_f - \rho_f v$ $H' = H - v \times D$ $P' = P$	$n \times (E^a - E^b) = 0$ $n \cdot (D^a - D^b) = \sigma_f$ $n \cdot (J_f^a - J_f^b) + \nabla_\Sigma \cdot K_f = v_n (\rho_f^a - \rho_f^b) - \frac{\partial \sigma_f}{\partial t}$ $n \times (H^a - H^b) = K_f + v_n n \times [n \times (D^a - D^b)]$

VI. Faraday's Disk (Homopolar Generator)



(a) A conducting disk rotating in an axial magnetic field is called a homopolar generator. (b) In addition to Ohmic and inductive voltages there is a speed voltage contribution proportional to the speed of the disk and the magnetic field.

$$B_0 = \frac{\mu_0 N i_f}{s}$$

$$\bar{J} = \sigma\left(\bar{E} + \bar{v}\times\bar{B}\right) \Rightarrow \bar{E} = \frac{\bar{J}}{\sigma} - \bar{v}\times\bar{B} \Rightarrow E_r = \frac{i_r}{2\pi\sigma\, dr} - \omega r B_0$$

$$\oint_L \bar{E}\cdot\bar{dl} = \int_1^2 E_r\, dr + \underbrace{\int_3^4 \bar{E}\cdot\bar{dl}}_{-v_r} = 0$$

$$v_r = \int_1^2 E_r\, dr = \int_{R_i}^{R_0}\left(\frac{i_r}{2\pi\sigma\, dr} - \omega r B_0\right)dr = \frac{i_r}{2\pi\sigma\, d}\ \ln\frac{R_0}{R_i} - \frac{\omega B_0}{2}\left(R_0^2 - R_i^2\right)$$

$$= i_r R_r - G\,\omega\, i_f$$

$$R_r = \frac{\ln \frac{R_0}{R_i}}{2\pi\sigma d}\ , \quad G = \frac{\mu_0 N}{2s}\left(R_0^2 - R_i^2\right)$$

Representative Numbers: copper $\left(\sigma \approx 6\times 10^7 \text{ Siemen}/\text{m}\right)$, $d = 1\,\text{mm}$

$\omega = 3600\,\text{rpm} = 120\,\pi\,\text{rad}/\text{s}$

$R_0 = 10\,\text{cm},\ R_i = 1\,\text{cm},\ B_0 = 1\,\text{tesla}$

$$v_{0c} = \frac{-\omega B_0}{2}\left(R_0{}^2 - R_i{}^2\right) \approx -1.9\,\text{V}$$

$$i_{sc} = \frac{v_{0c}\,2\pi\sigma d}{\ln\left(\frac{R_0}{R_i}\right)} \approx 3\times 10^5\ \text{amp}$$

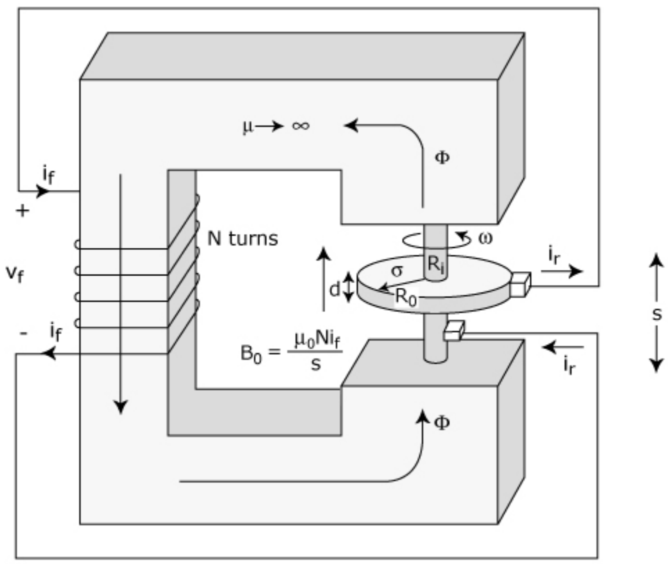
$$T = \int_{\phi=0}^{2\pi}\int_{z=0}^{d}\int_{r=R_i}^{R_0} r\,\bar{i}_r \times \left(\bar{J}\times\bar{B}\right) r\,dr\,d\phi\,dz$$

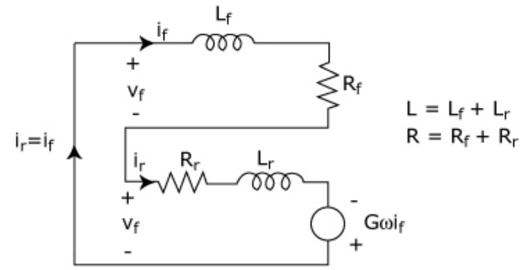
$$= -i_r B_0\,\bar{i}_z \int_{R_i}^{R_0} r\,dr$$

$$= \frac{-i_r B_0}{2}\left(R_0{}^2 - R_i{}^2\right)\bar{i}_z$$

$$= -G\,i_f\,i_r\,\bar{i}_z$$

VII. Self-Excited DC Homopolar Generator





A homopolar generator can be self-excited where the generated rotor current is fed back to the field winding to generate its own magnetic field.

$$i_f = i_r \equiv i$$

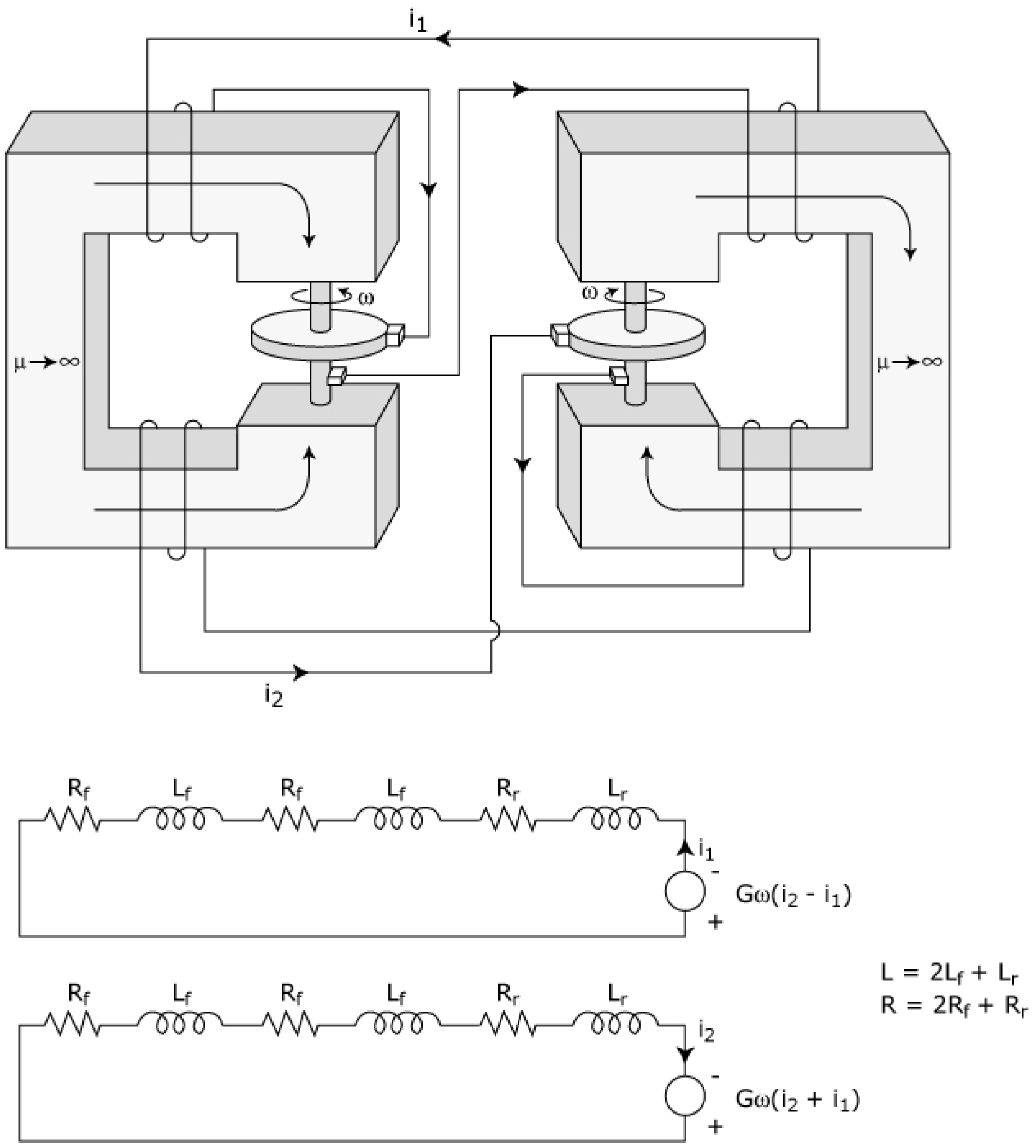
$$L\frac{di}{dt} + i(R - G\omega) = 0\ ; \quad R = R_r + R_f$$

$$L = L_r + L_f$$

$$i(t) = I_0\, e^{-[R - G\omega]t/L}$$

$G\omega > R$ Self-Excited

VIII. Self-Excited AC Homopolar Generator



Cross-connecting two homopolar generators can result in self-excited two-phase alternating currents. Two independent field windings are required where on one machine the fluxes add while on the other they subtract.

$$L\frac{di_1}{dt} + (R - G\omega)i_1 + G\omega i_2 = 0$$

$$L\frac{di_2}{dt} + (R - G\omega)i_2 - G\omega i_1 = 0$$

$i_1 = I_1 e^{st}$, $i_2 = I_2 e^{st}$

$$(Ls + R - G\omega) I_1 + G\omega I_2 = 0$$

$$-G\omega I_1 + (Ls + R - G\omega) I_2 = 0$$

$$(Ls + R - G\omega)^2 + (G\omega)^2 = 0$$

$$Ls + R - G\omega = \pm jG\omega$$

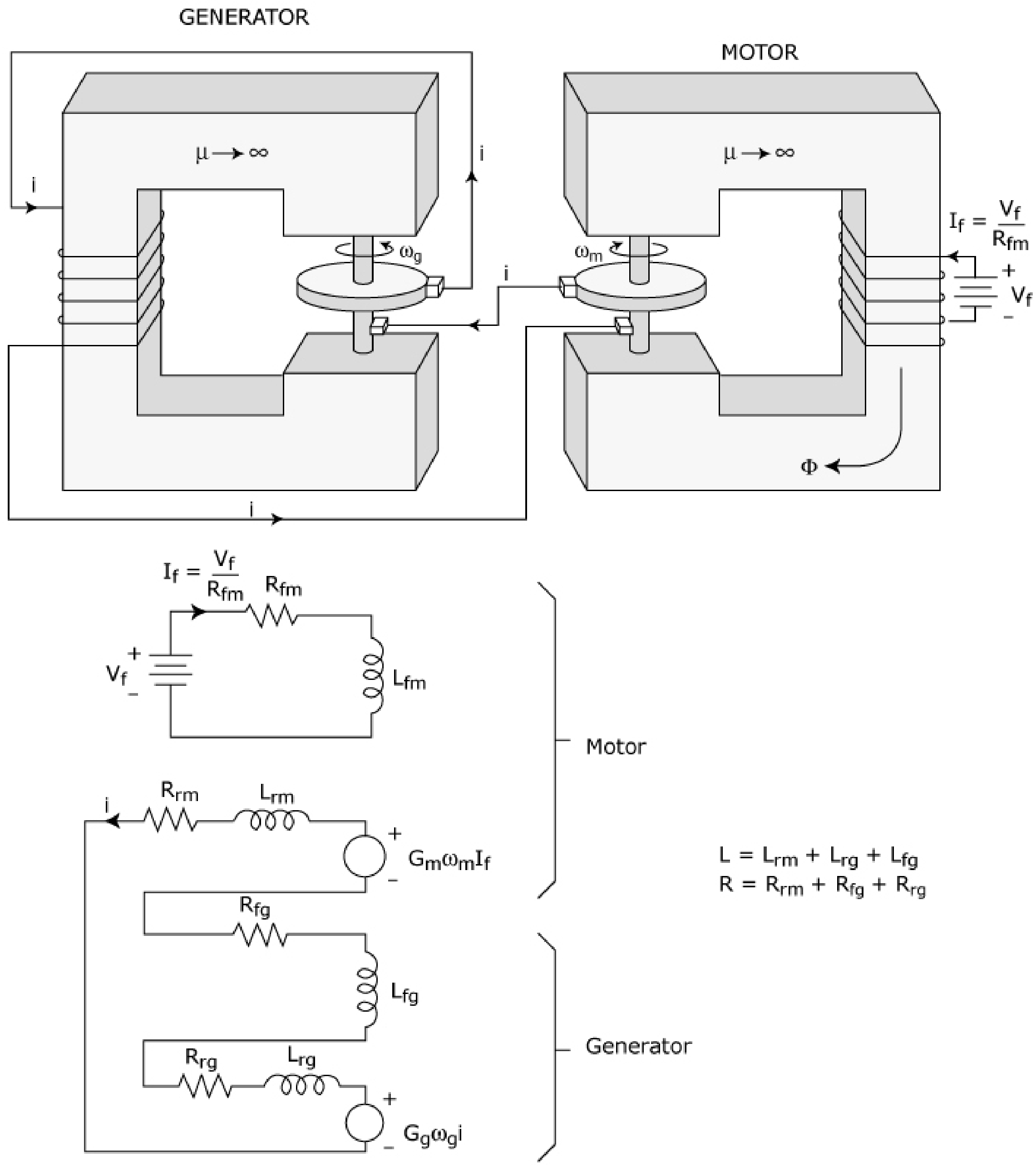
$$s = -\frac{(R - G\omega)}{L} \pm j\frac{G\omega}{L}$$

$$\frac{I_1}{I_2} = \frac{-G\omega}{(Ls + R - G\omega)} = \pm j$$

Self Excited: $G\omega > R$

Oscillation frequency: $\omega_0 = \text{Im}(s) = G\omega/L$

IX. Self-Excited Periodic Motor Speed Reversals



Cross-connecting a homopolar generator and motor can result in spontaneous periodic speed reversals of the motor's shaft.

$$\frac{di}{dt} + \frac{\left(R - G_g\,\omega_g\right)i}{L} = \frac{G_m\,\omega_m\,I_f}{L}$$

$$J\frac{d\omega_m}{dt} = -G_m I_f i$$

$$i = I e^{st}, \omega_m = W e^{st}$$

$$I\left[s + \frac{R - G_g \omega_g}{L}\right] - W\left(\frac{G_m I_f}{L}\right) = 0$$

$$I\left(\frac{G_m I_f}{J}\right) + Ws = 0$$

$$s\left[s + \frac{R - G_g \omega_g}{L}\right] + \frac{(G_m I_f)^2}{JL} = 0$$

$$s = -\frac{(R - G_g \omega_g)}{2L} \pm \left[\left(\frac{R - G_g \omega_g}{2L}\right)^2 - \frac{(G_m I_f)^2}{JL}\right]^{1/2}$$

Self-excitation: $G_g \omega_g > R$

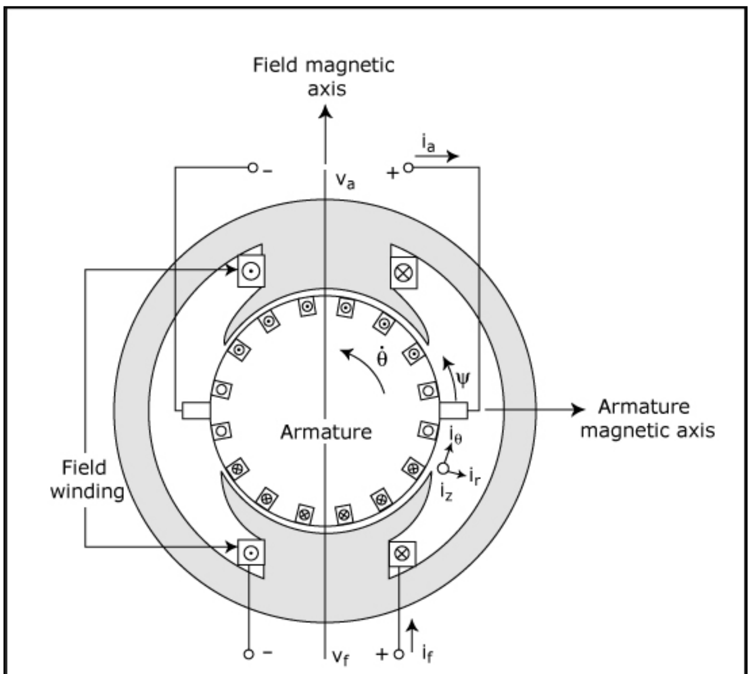
Oscillations if s has an imaginary part:

$$\frac{(G_m I_f)^2}{JL} > \left(\frac{R - G_g \omega_g}{2L}\right)^2$$

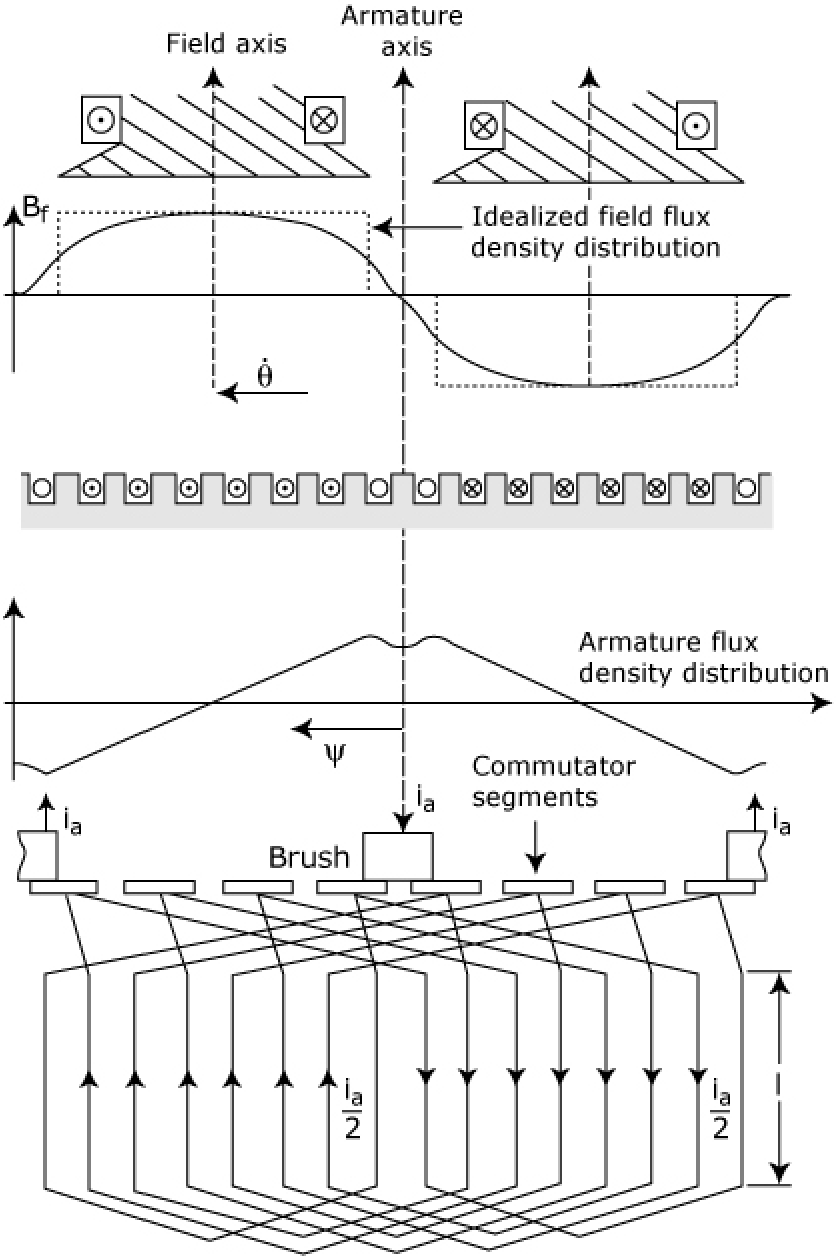
X. DC Commutator Machines

Quasi-One Dimensional Description

A. Electrical Equations



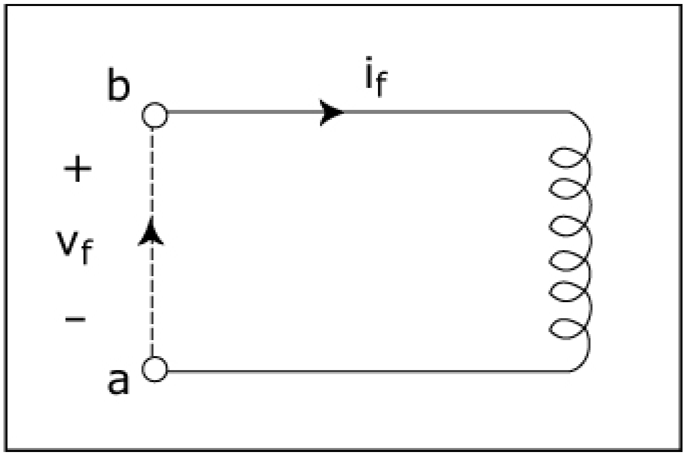
Two-pole, commutator machine.



Developed views of two-pole commutator machine showing flux distributions and armature connections.

$$\oint_C \bar{E} \cdot \overline{dl} = -\frac{d}{dt}\int_S \bar{B} \cdot \bar{n}\, da$$

1. Field Winding



$$\oint_C \bar{E} \cdot \overline{dl} = -v_f + \int_{winding} \underbrace{\frac{i_f}{A\sigma}}_{\frac{J}{\sigma}} dl = -v_f + i_f \underbrace{R_f}_{\text{Resistance of field winding}}$$

$$\lambda_f = \int_S \bar{B} \cdot \bar{n}\, da = L_f\, i_f$$

$$-v_f + i_f R_f = -L_f \frac{di_f}{dt}$$

$$v_f = L_f \frac{di_f}{dt} + i_f R_f$$

2. Armature Winding

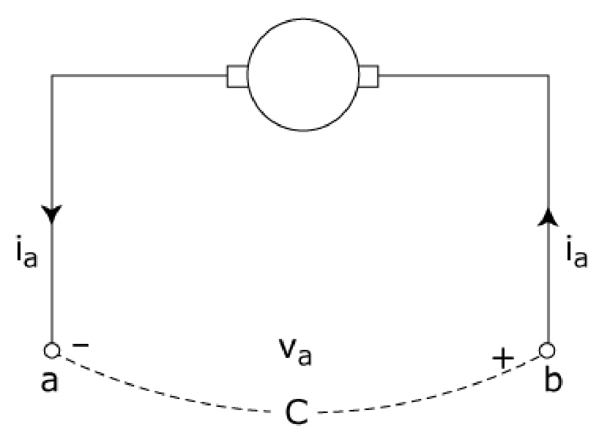


Illustration of the contour for finding an armature voltage equation. The contour is completed on the armature, where it follows one of the two conducting paths joining the brushes in figure on page 16.

Reminder: $\overline{f} = q\left(\overline{E} + \overline{v} \times \overline{B}\right) = q\overline{E}'$

$\overline{E}' = \overline{E} + \overline{v} \times \overline{B}$

Take Stationary Contour through armature winding

$\overline{E} = \overline{E}' - \overline{v} \times \overline{B}$

$$\oint_C \overline{E} \bullet \overline{dl} = -v_a + \int \left(\overline{E}' - \overline{v} \times \overline{B}\right) \bullet \overline{dl}$$

$$= -v_a + \int_a^b \left(\frac{i_a}{A\sigma} + \omega R B_r\, \overline{i}_z\right) \bullet \overline{dl} \quad ; \qquad v = \omega R\, \overline{i}_\theta$$

$$\overline{B} = \overline{i}_r B_r(\chi)$$

$$= -v_a + i_a R_a + \omega R \left(B_{rf}\right)_{av} lN$$

$$= -\frac{d}{dt}\int_S \overline{B} \bullet \overline{da} = -L_a \frac{di_a}{dt}$$

$$v_a = i_a R_a + L_a \frac{di_a}{dt} + G\omega i_f \qquad \left(G i_f = lNR\left(B_{rf}\right)_{av}\right)$$

B. Mechanical Equations

$$\overline{F} = \overline{i}_\theta J_z B_r = \overline{i}_\theta \frac{i_a}{A_w} B_r, \; \overline{f} = \overline{F} A_w l = \overline{i}_\theta i_a l B_r$$

$$T = fR = i_a l B_r R N = G i_f i_a$$

$$J\frac{d^2\theta}{dt^2} = T = G i_f i_a$$

C. Linear Amplifier

1) Open Circuit

$$v_f = V_f,\; i_a = 0 \Rightarrow i_f = V_f/R_f$$

$$v_a = G\omega V_f/R_f$$

2) Resistively Loaded Armature (DC Generator)

$$v_a = -i_a R_L = i_a R_a + G\omega V_f/R_f$$

$$i_a = \frac{-G\,\omega\,V_f}{R_f\,(R_a + R_L)}$$

$$v_a = \frac{G\,\omega\,V_f\,R_L}{R_f\,(R_a + R_L)}$$

D. DC Motors

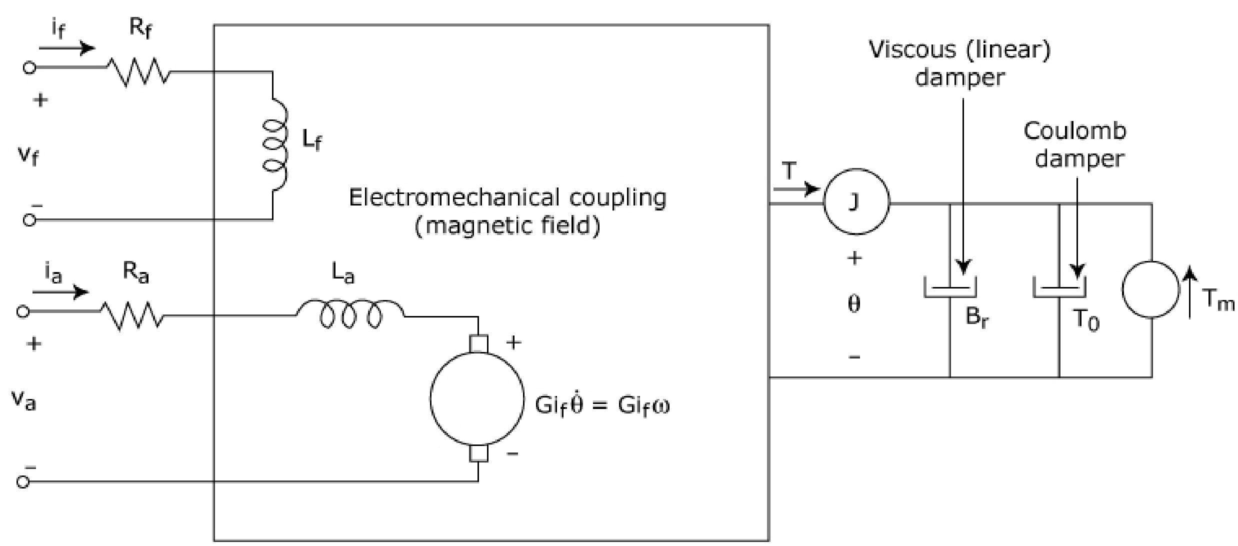
1) Shunt Excitation: $v_a = v_f = v_t$

$$v_t = i_f\,R_f = i_a\,R_a + G\,\omega\,i_f$$

$$i_f\,(R_f - G\omega) = i_a\,R_a$$

$$i_f = \frac{V_t}{R_f},\; i_a = \frac{V_t}{R_f}\frac{(R_f - G\omega)}{R_a}$$

$$T = G\,i_f\,i_a = G\left(\frac{V_t}{R_f}\right)^2 \frac{(R_f - G\omega)}{R_a}$$



Equivalent circuit of the commutator machine in figure on page 15.

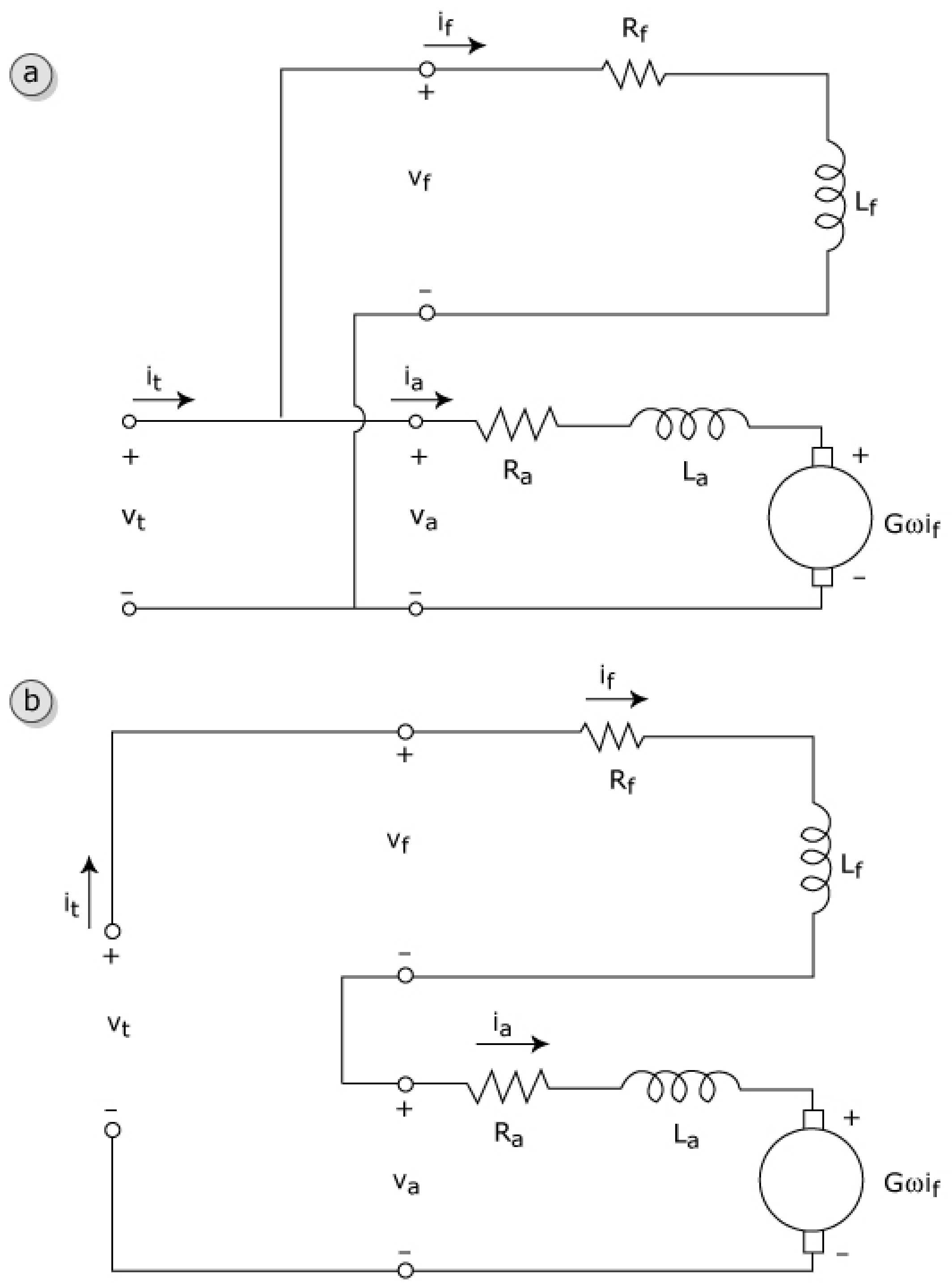
2) Series: $i_a = i_f = i_t$

$$i_t \left(R_f + R_a + G\omega\right) = v_t$$

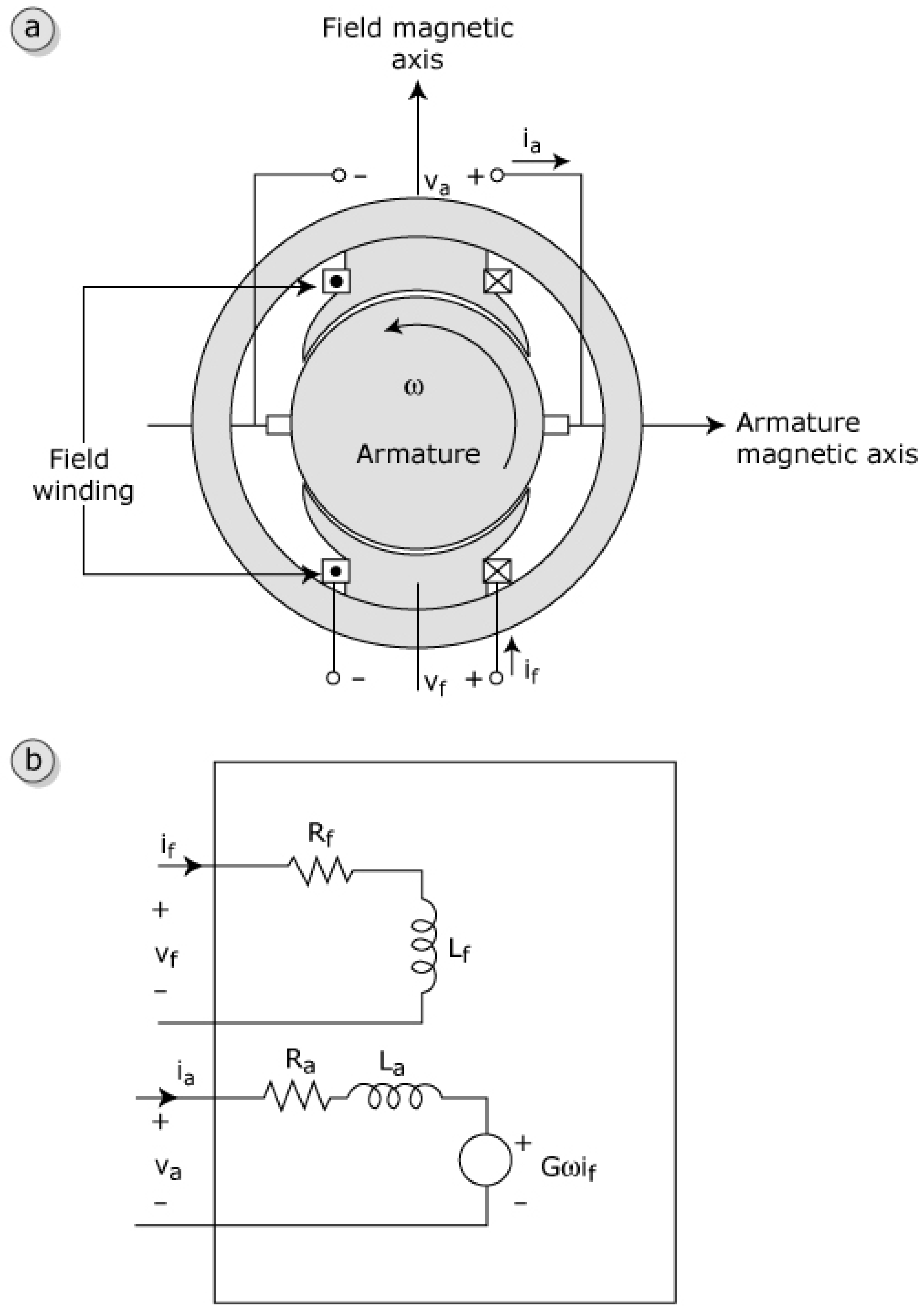
$$i_t = \frac{v_t}{\left(R_f + R_a + G\omega\right)}$$

$$T = G i_t^2 = G \frac{v_t^2}{\left(R_f + R_a + G\omega\right)^2}$$

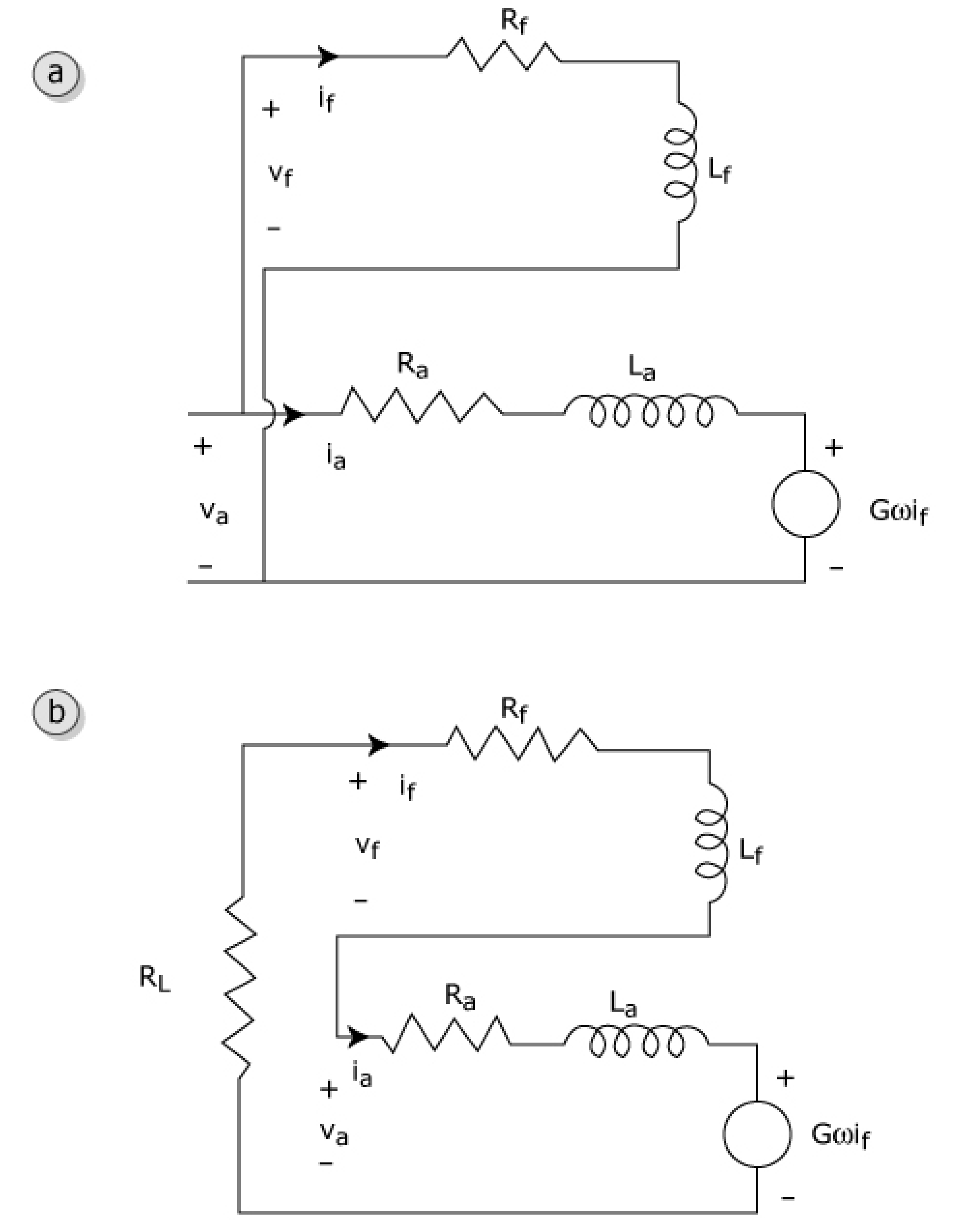
XI. Self-Excited Machines



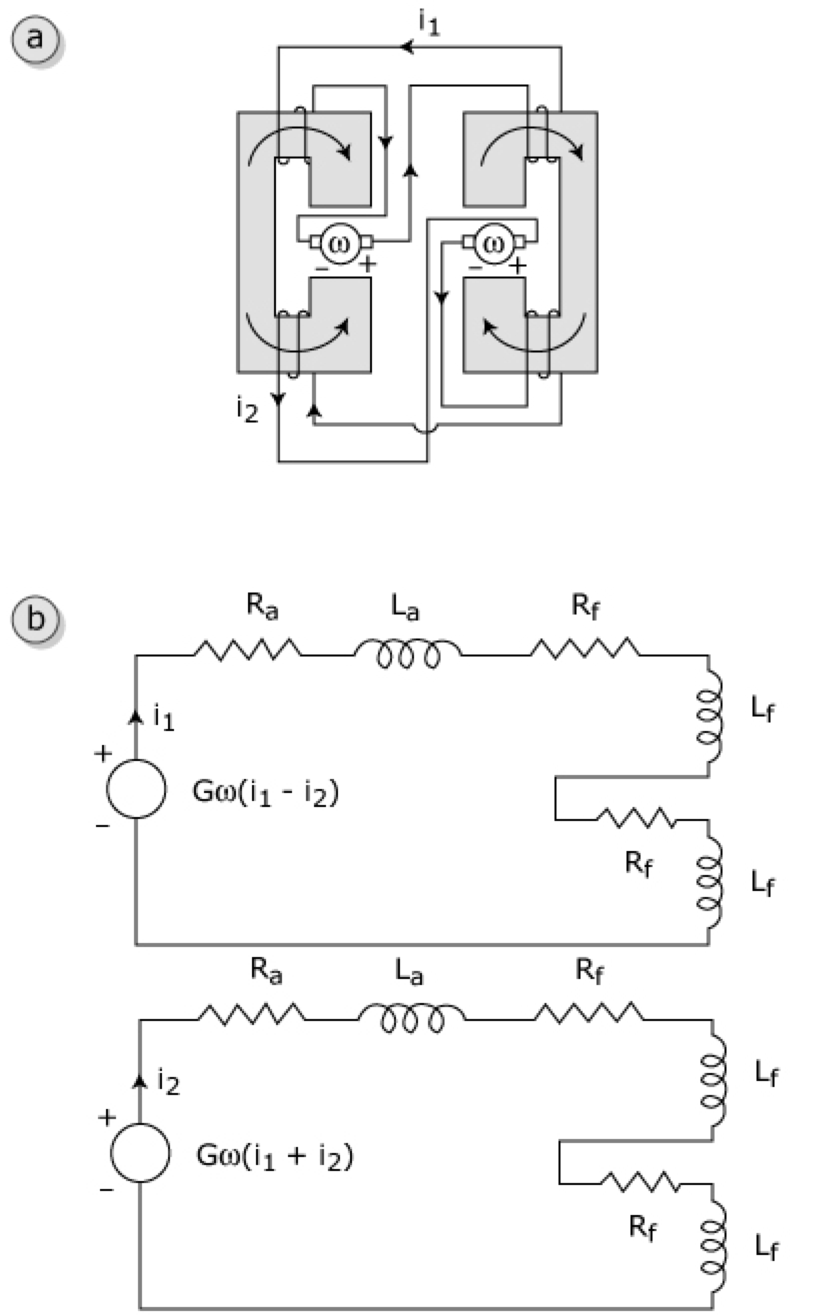
Methods of self-exciting a dc motor: (a) shunt excitation; (b) series excitation.



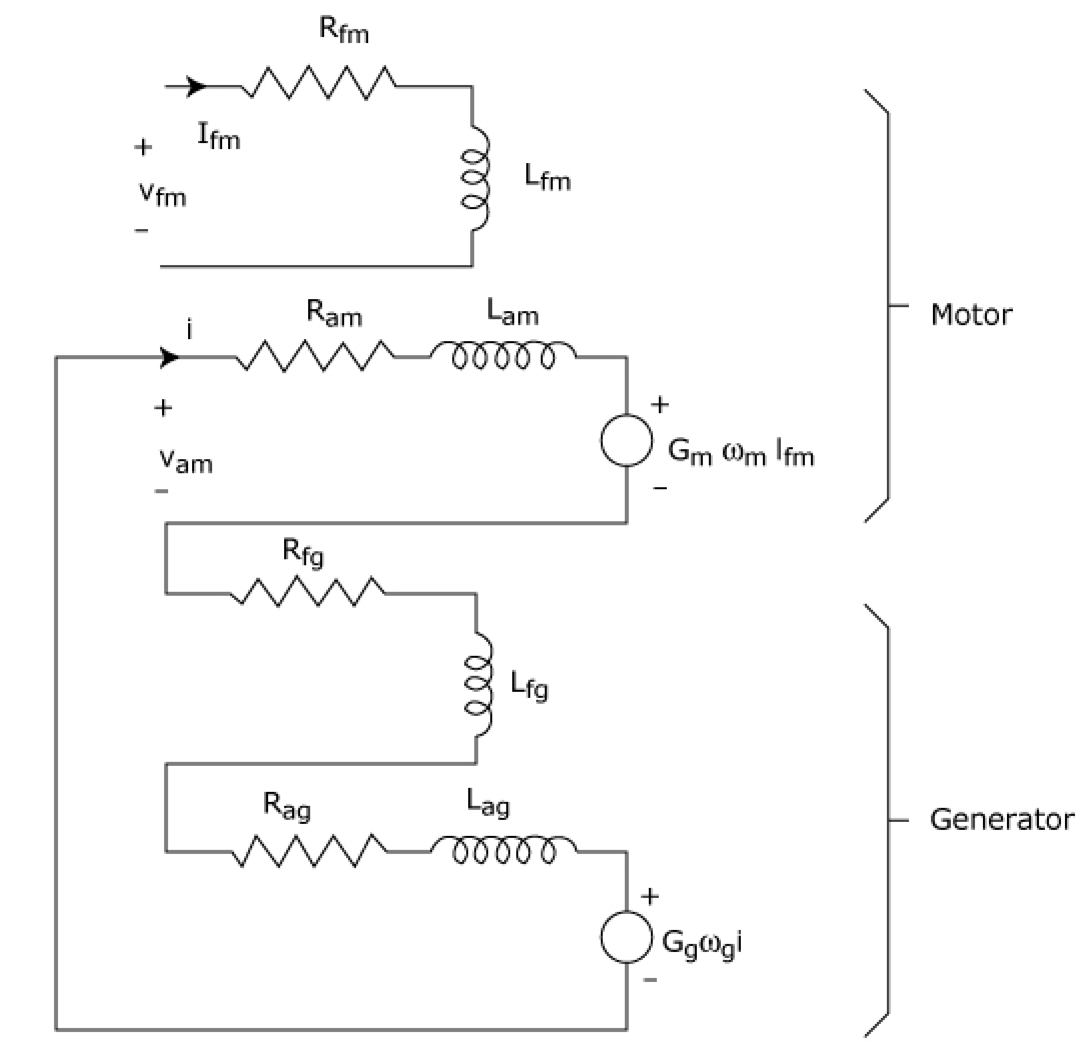
Two-pole commutator machine. (a) Geometry. (b) Equivalent circuit.



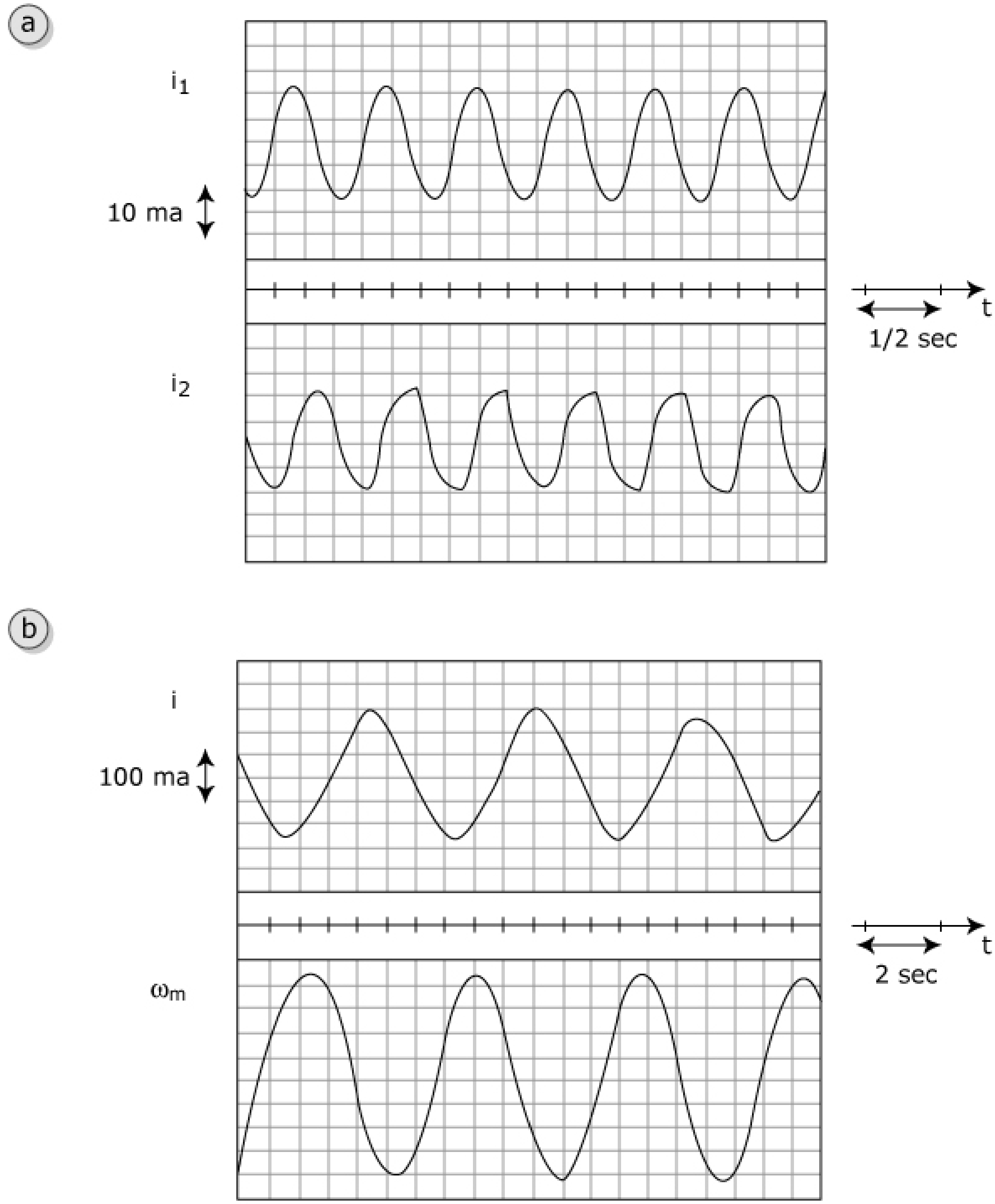
Equivalent circuit models for shunt and series self-excited generators. (a) Open-circuit shunt. (b) Series with load resistor.



Configuration for obtaining a.c. power from two identical generators.
(a) Wiring configuration. (b) Equivalent circuit description.



Equivalent circuit description to obtain spontaneous electrical and mechanical oscillations.



(a) Two-phase currents obtained from a pair of coupled d.c. machines rotating at a speed of 1790 rev/min. (b) Alternating current and speed for the coupled motor-generator combination with $I_f = 0.15$ A and generator shaft speed of 1620 rev/min.