6.641 Electromagnetic Fields, Forces, and Motion
Spring 2009

6.641 — Electromagnetic Fields, Forces, and Motion		Spring 2006
	Quiz 2 - Solutions	
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Problem 1

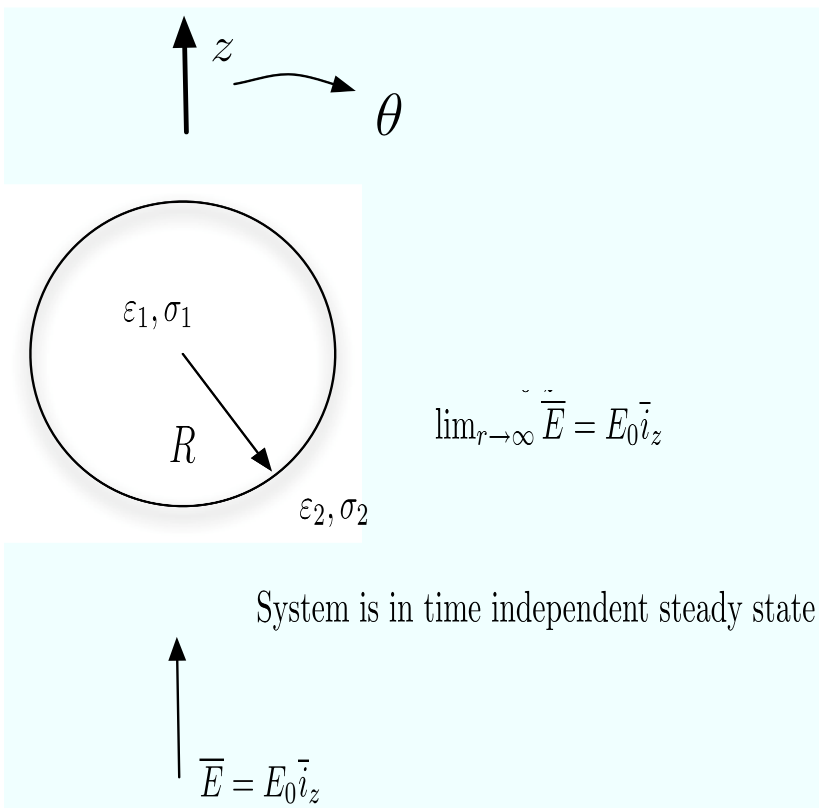

Figure 1: Lossy dielectric sphere within a lossy dielectric with an imposed uniform electric field $\overline{E} = E_o\overline{i_z}$. (Image by MIT OpenCourseWare.)

A

Question: What are the necessary boundary conditions to solve for the electrostatic scalar potential $\Phi(r,\theta)$ and electric field $\overline{E}(r\theta)$ inside and outside the sphere in the time independent steady state?

Solution:

$$\Phi(r = 0) \to \text{finite}$$

$$\Phi(r = R_-) = \Phi(r = R_+)$$

$$J_r(r = R_-) = J_r(r = R_+) \Rightarrow \sigma_1 E_r(r = R_-) = \sigma_2 E_r(r = R_+)$$
$$\Phi(r \to \infty) = -E_0 z = -E_0 r \cos\theta$$

B

Question: Find $\Phi(r,\theta)$ and $\overline{E}(r,\theta)$ in the time independent steady state.

Solution:

$$\Phi = R(r)F(\theta)$$

Solution for $n = 1$ case: $R(r) = Ar + B\frac{1}{r^2}$, $F(\theta) = C\cos\theta$.

$$\Phi = \begin{cases} \left(Ar + B\frac{1}{r^2}\right)\cos\theta & r < R \\ \left(Cr + D\frac{1}{r^2}\right)\cos\theta & r > R \end{cases}$$

$$\begin{aligned} &\text{BC I} && \Phi(r=0)\text{finite} \Rightarrow B = 0 \\ &\text{BC IV} && \Phi(r\to\infty) = -E_0 r\cos\theta \Rightarrow C = -E_0 \\ &\text{BC II} && \Phi(r = R_-) = \Phi(r = R_+) \Rightarrow AR = -E_0 R + D\frac{1}{R^2} \\ &\text{BC III} && J_r(r = R_-) = J_r(r = R_+) \Rightarrow -\sigma_1\frac{\partial}{\partial r}\Phi(r = R_-) = -\sigma_2\frac{\partial}{\partial r}\Phi(r = R_+) \Rightarrow \sigma_1 A = +\sigma_2\left[-E_0 - \frac{2D}{R^3}\right] \end{aligned}$$

$$-E_0 + D\frac{1}{R^3} = \frac{\sigma_2}{\sigma_1}\left[-E_0 - \frac{2D}{R^3}\right] = -E_0\frac{\sigma_2}{\sigma_1} - \frac{2}{R^3}\frac{\sigma_2}{\sigma_1}D$$

$$D\left[\frac{1}{R^3} + \frac{2\sigma_2}{\sigma_1}\frac{1}{R^3}\right] = E_0 - \frac{\sigma_2}{\sigma_1}E_0 \Rightarrow D = \frac{R^3 E_0\left(1 - \frac{\sigma_2}{\sigma_1}\right)}{\left(1 + \frac{2\sigma_2}{\sigma_1}\right)} \Rightarrow D = R^3 E_0\frac{\sigma_1 - \sigma_2}{\sigma_1 + 2\sigma_2}$$

$$\Rightarrow A = -E_0 + E_0\left(\frac{\sigma_1 - \sigma_2}{\sigma_1 + 2\sigma_2}\right) = E_0\left(\frac{\sigma_1 - \sigma_2 - \sigma_1 - 2\sigma_2}{\sigma_1 + 2\sigma_2}\right) = -E_0\frac{3\sigma_2}{\sigma_1 + 2\sigma_2} = A$$

$$\Rightarrow \Phi = \begin{cases} -\frac{3\sigma_2 E_0}{2\sigma_2 + \sigma_1} r\cos\theta & r < R \\ -E_0\left(r - \frac{(\sigma_1 - \sigma_2)R^3}{(2\sigma_2 + \sigma_1)r^2}\right)\cos\theta & r > R \end{cases}$$

$$\overline{E} = -\nabla\Phi = -\left[\frac{\partial\Phi}{\partial r}\bar{i}_r + \frac{1}{r}\frac{\partial\Phi}{\partial\theta}\bar{i}_\theta\right]$$

$$\overline{E} = \begin{cases} \frac{3\sigma_2 E_0}{2\sigma_2 + \sigma_1}\left(\cos\theta\bar{i}_r - \sin\theta\bar{i}_\theta\right) & r < R \\ E_0\left[1 + \frac{2(\sigma_1 - \sigma_2)R^3}{(2\sigma_2 + \sigma_1)r^3}\right]\cos\theta\bar{i}_r - E_0\left[1 - \frac{(\sigma_1 - \sigma_2)R^3}{(2\sigma_2 + \sigma_1)r^3}\right]\sin\theta\bar{i}_\theta & r > R \end{cases}$$

C

Question: What is the free surface charge distribution on the $r = R$ interface?

Solution:

$$\sigma_f = \varepsilon_2 E_r(r = R_+) - \varepsilon_1 E_r(r = R_-)$$

$$\begin{aligned}
&= \varepsilon_2 E_0 \left[1 + \frac{2(\sigma_1 - \sigma_2)R^3}{(2\sigma_2 + \sigma_1)R^3}\right]\cos\theta - \varepsilon_1 E_0 \frac{3\sigma_2}{2\sigma_2 + \sigma_1}\cos\theta \\
&= \varepsilon_2 E_0 \left[\frac{2\sigma_2 + \sigma_1 + 2\sigma_1 - 2\sigma_2}{2\sigma_2 + \sigma_1}\right]\cos\theta - \varepsilon_1 E_0 \frac{3\sigma_2}{2\sigma_2 + \sigma_1}\cos\theta \\
&= E_0 \cos\theta \left[\frac{3\sigma_1\varepsilon_2 - 3\sigma_2\varepsilon_1}{2\sigma_2 + \sigma_1}\right]
\end{aligned}$$

$$\sigma_f = 3E_0 \cos\theta \frac{\varepsilon_2\sigma_1 - \varepsilon_1\sigma_2}{2\sigma_2 + \sigma_1} \text{ at } r = R \text{ interface}$$

D

Question: For what relationship between $\varepsilon_1, \varepsilon_2, \sigma_1$, and σ_2 is the free surface charge density zero for all θ on the $r = R$ interface?

Solution: For $\varepsilon_2\sigma_1 - \varepsilon_1\sigma_2 = 0$, $\sigma_f = 0$. Therefore, $\frac{\varepsilon_2}{\sigma_2} = \frac{\varepsilon_1}{\sigma_1}$.

E

Question: What is the effective dipole moment of the sphere for fields in the region $r > R$?

Solution: For dipole, the potential has the form

$$\Phi = \frac{p\cos\theta}{4\pi\varepsilon_2 r^2} = \frac{E_0(\sigma_1 - \sigma_2)R^3}{(2\sigma_2 + \sigma_1)r^2}\cos\theta$$

$$\Rightarrow p = 4\pi\varepsilon_2 E_0 R^3 \frac{(\sigma_1 - \sigma_2)}{(2\sigma_2 + \sigma_1)}$$

p is effective dipole moment.

Problem 2

$$i_1 = \frac{\lambda_1 \left[L_0 - M\cos 2\theta\right] - \lambda_2 M \sin 2\theta}{L_0^2 - M^2}$$

$$i_2 = \frac{-\lambda_1 M \sin 2\theta + \lambda_2(L_0 + M\cos 2\theta)}{L_0^2 - M^2} \qquad L_0 > M$$

A

<u>Question:</u> Determine the magnetoquasistatic torque $T^M(\lambda_1, \lambda_2, \theta)$.

$$dW(\lambda_1, \lambda_2, \theta) = i_1 d\lambda_1 + i_2 d\lambda_2 - T^m d\theta \qquad L_0 > M$$

<u>Solution:</u>

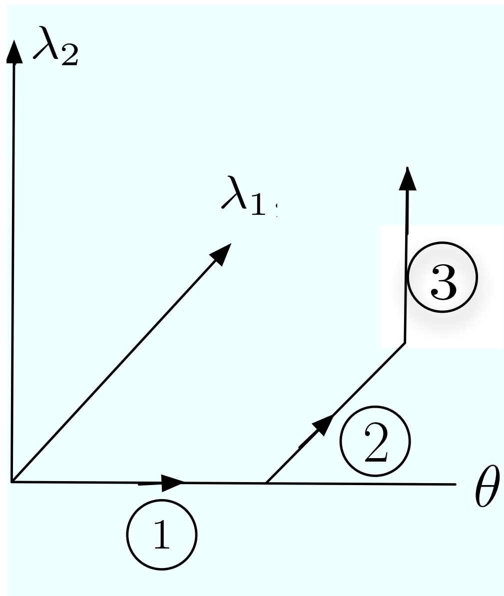

Figure 2: Line integral path in $(\lambda_1, \lambda_2, \theta)$ space to compute magnetic energy W and magnetic torque T^M (Image by MIT OpenCourseWare.)

$$W = \underbrace{\int}_{\substack{\boxed{1} \\ \lambda_1=\lambda_2=0}} \cancelto{0}{T^M} d\theta + \underbrace{\int}_{\substack{\boxed{2} \\ \lambda_2=0 \\ \theta=\text{constant}}} i_1 d\lambda_1 + \underbrace{\int}_{\substack{\boxed{3} \\ \lambda_1=\text{constant} \\ \theta=\text{constant}}} i_2 d\lambda_2$$

$$\boxed{W = \frac{\lambda_1^2}{2}\frac{L - M\cos 2\theta}{L_0^2 - M^2} - \lambda_1\lambda_2 \frac{M\sin 2\theta}{L_0^2 - M^2} + \frac{\lambda_2^2}{2}\frac{L_0 + M\cos 2\theta}{L_0^2 - M^2}}$$

$$\begin{aligned} T^M = -\left.\frac{dW}{d\theta}\right|_{\lambda_1, \lambda_2} &= \frac{-\lambda_1^2 M(-2\sin 2\theta)}{2(L_0^2 - M^2)} - \frac{\lambda_1\lambda_2 M 2\cos 2\theta}{L_0^2 - M^2} + \frac{\lambda_2^2 M(-2\sin 2\theta)}{2(L_0^2 - M^2)} \\ &= \frac{M}{L_0^2 - M^2}\left[\lambda_1^2 \sin 2\theta - 2\lambda_1\lambda_2\cos 2\theta - \lambda_2^2 \sin 2\theta\right] \end{aligned}$$

$$\boxed{T^M = \frac{M}{L_0^2 - M^2}\left[(\lambda_1^2 - \lambda_2^2)\sin 2\theta - 2\lambda_1\lambda_2 \cos 2\theta\right]}$$

B

Question: Assume that the machine is excited by voltage sources such that $V_1 = \frac{d\lambda_1}{dt} = V_0 \cos \omega t$, $V_2 = \frac{d\lambda_2}{dt} = V_0 \sin \omega t$, and the rotor has the constant angular velocity ω_m such that $\theta = \omega_m t + \gamma$. What are λ_1 and λ_2 as a sinusoidal steady state function of time? Evaluate the instantaneous torque T^M. Under what conditions is it constant?

Solution:

$$V_1 = \frac{d\lambda_1}{dt} = V_0 \cos \omega t, \qquad V_2 = \frac{d\lambda_2}{dt} = V_0 \sin \omega t$$

$$\Rightarrow \lambda_1 = \frac{V_0}{\omega} \sin \omega t, \lambda_2 = -\frac{V_0}{\omega} \cos \omega t$$

$$T^M = \frac{M}{L_0^2 - M^2} \left[\left(\frac{V_0^2}{\omega^2} \left(\sin^2 \omega t - \cos^2 \omega t \right) \sin 2\theta + \frac{V_0^2}{\omega^2} 2 \sin \omega t \cos \omega t \cos 2\theta \right) \right]$$

$$= \frac{M}{(L_0^2 - M^2)} \frac{V_0^2}{\omega^2} \left[-\cos 2\omega t \sin 2\theta + \sin 2\omega t \cos 2\theta \right]$$

$$= \frac{M}{L_0^2 - M^2} \frac{V_0^2}{\omega^2} \left[\sin 2(\omega t - \theta) \right] \qquad \theta = \omega_m t + \gamma$$

$$\Rightarrow T^M = \frac{M}{L_0^2 - M^2} \frac{V_0^2}{\omega^2} \sin \left[2(\omega - \omega_m) t - 2\gamma \right]$$

For $\omega = \omega_m$, T^M is constant.

C

Question: The rotor is subject to a mechanical torque (acting on it in the $+\theta$-direction): $T = T_0 + T'(t)$, where T_0 is a positive constant. The time-varying part of the torque perturbs the steady rotation of (b) so that $\theta = \omega_m t + \gamma_0 + \gamma'(t)$. Assume that the rotor has a moment of inertia J but that there is no damping. Find the possible equilibrium angles γ_0 between the rotor and the stator field and indicate which are stable and unstable. Then write a differential equation for $\gamma'(t)$, with $T'(t)$ as a driving function.

Solution:

$$J \frac{\partial^2 \theta}{\partial t^2} = T_{\text{total}} = T_{\text{mechanical}} + T_{\text{electrical}}$$

$$J \frac{\partial^2 \theta}{\partial t^2} = T_0 + T'(t) + T^M$$

For steady state equilibrium

$$0 = T_0 + T^M \Rightarrow T_0 = -T^M \Rightarrow \boxed{T_0 = \frac{M}{L_0^2 - M^2} \left(\frac{V_0}{\omega} \right)^2 \sin 2\gamma_0}$$

In steady state,

$$T_T = T^M + T_0 = -\frac{M}{L_0^2 - M^2} \left(\frac{V_0}{\omega} \right)^2 \sin 2\gamma_0 + T_0 = 0$$

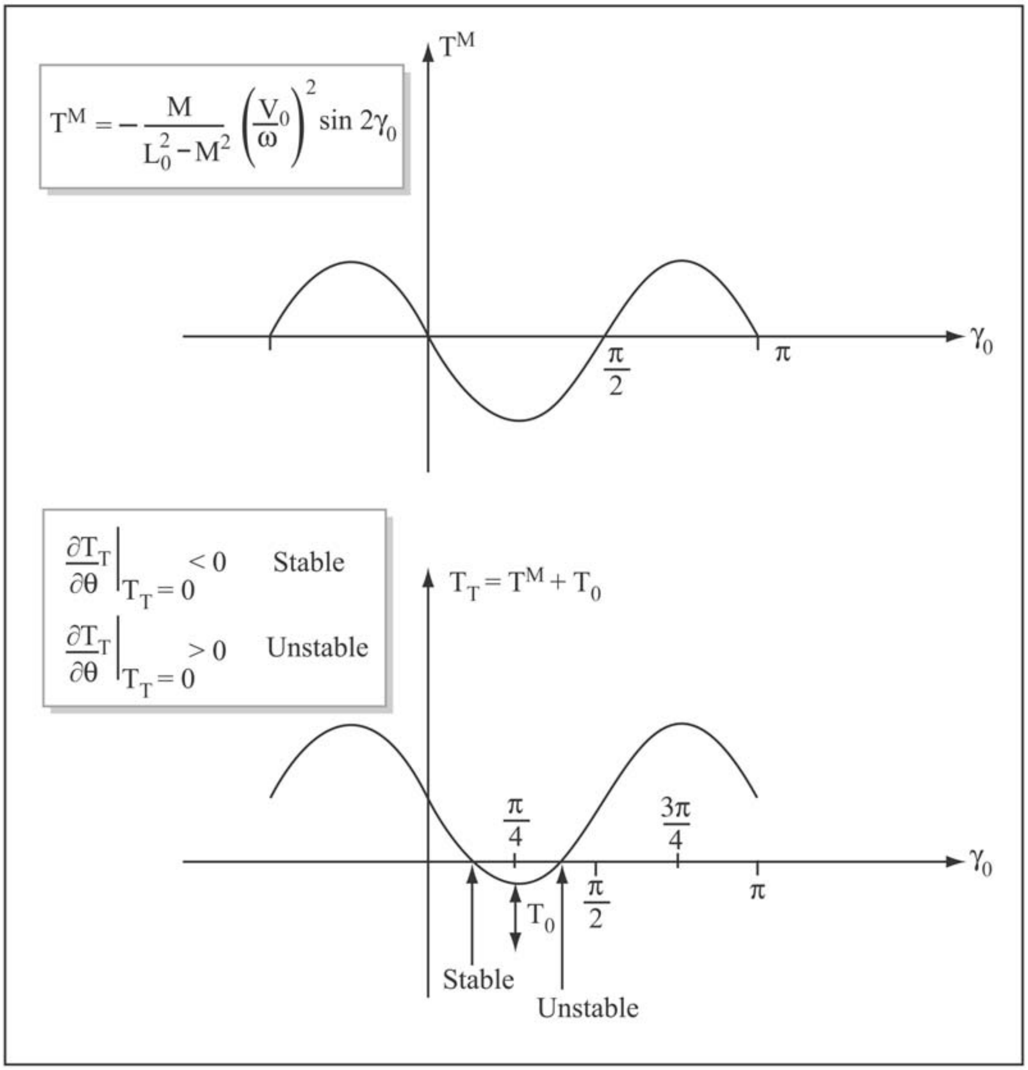

Figure 3: Magnetic and total (magnetic plus mechanical) torques. The equilibrium angle for γ_0 has a total torque equal to zero with stable and unstable solutions (Image by MIT OpenCourseWare.)

$$\frac{\partial T_T}{\partial \theta}\Big|_{T_T=0} < 0 \quad \text{stable}$$

$$\frac{\partial T_T}{\partial \theta}\Big|_{T_T=0} > 0 \quad \text{unstable}$$

D

Question: Consider small perturbations of the rotation $\gamma'(t)$, so that the equation of motion found in (c) can be linearized. Find the response to an impulse of torque $T'(t) = I_0\delta(t)$, assuming that before the impulse in torque the rotation velocity is constant at ω_m.

Solution:

$$T_T(\gamma) = T_0 + T' - \frac{M}{(L_0^2 - M^2)}\left(\frac{V_0}{\omega}\right)^2 \sin 2\gamma$$

Taylor expanding about equilibrium angle

$$\begin{aligned}
T_T &\approx T_T(\gamma_{\text{eq}}) + \frac{\partial T_T}{\partial \gamma}\Big|_{\gamma_{\text{eq}}} \gamma' + T' \\
&\approx \underbrace{T_0 - \frac{M}{L_0^2 - M^2}\frac{V_0^2}{\omega^2}\sin 2\gamma_{\text{eq}}}_{0} + T' - \left[\frac{2M}{L_0^2 - M^2}\frac{V_0^2}{\omega^2}\cos 2\gamma_{\text{eq}}\right]\gamma' \\
&\approx T' - \left[\frac{2M}{L_0^2 - M^2}\left(\frac{V_0}{\omega}\right)^2 \cos 2\gamma_{\text{eq}}\right]\gamma' \\
&\Rightarrow \boxed{J\frac{d^2\gamma'}{dt^2} + \left[\frac{2M}{L_0^2 - M^2}\left(\frac{V_0}{\omega}\right)^2 \cos 2\gamma_{\text{eq}}\right]\gamma' = T'}
\end{aligned}$$

(Linearized perturbation equation)

$$\frac{d^2\gamma'}{dt^2} + \omega_0^2 \gamma' = \frac{T'}{J}$$

where

$$\omega_0^2 = \frac{2M}{J(L_0^2 - M^2)}\left(\frac{V_0}{\omega}\right)^2 \cos 2\gamma_{\text{eq}}$$

For $T'(t) = I_0\delta(t)$, the differential equation to solve is

$$\frac{d^2\gamma'}{dt^2} + \omega_0^2 \gamma' = \frac{I_0\delta(t)}{J}$$

Solution is in the form $\gamma'(t) = A\sin\omega_0 t + B\cos\omega_0 t$. Initial conditions for this impulse of torque would be

$$\frac{d\gamma'}{dt}\Big|_{t=0_+} = \frac{I_0}{J} \Rightarrow A\omega_0 = \frac{I_0}{J} \Rightarrow A = \frac{I_0}{J\omega_0}$$

$$\gamma'(t = 0_+) = 0 \Rightarrow B = 0$$

$$\Rightarrow \gamma'(t) = \frac{I_0}{J\omega_0}\sin\omega_0 t$$

E

Question: **Which of the equilibrium phase angles γ_0 found in (c) is stable? Verify the stability or instability of the equilibrium angles found in part (c) using the results of part (d).**

Solution: The differential equation

$$\frac{d^2\gamma'}{dt^2} + \omega_0^2 \gamma' = \frac{T'}{J}$$

will have sinusoidal solutions for $\omega_0^2 > 0$ and exponential solutions for $\omega_0^2 < 0$. Thus, for stability $\omega_0^2 > 0 \Rightarrow \cos 2\gamma_{\text{eq}} > 0 \Rightarrow$ the first equilibrium angle ($0 < \gamma_{\text{eq}} < \frac{\pi}{2}$) found in part (c) is stable and the other angle ($\frac{\pi}{2} < \gamma_{\text{eq}} < \pi$) is unstable.