6.641 Electromagnetic Fields, Forces, and Motion
Spring 2009

6.641 — Electromagnetic Fields, Forces, and Motion	Spring 2008
Final- Solutions	
Prof. Markus Zahn	MIT OpenCourseWare

Problem 1

A

Question: **What are the electric potential solutions in the regions $0 \leq y \leq a$ and $-b \leq y \leq 0$?**

Solution:

$$\Phi(x,y) = \begin{cases} -\frac{V_0}{\sinh ka} \cos kx \sinh k(y-a) & 0 \leq x \leq a \\ \frac{V_0}{\sinh kb} \cos kx \sinh k(y+b) & -b \leq x \leq 0 \end{cases}$$

B

Question: **What are the electric field distributions in the regions $0 < y < a$ and $-b < y < 0$?**

Solution:

$$\overline{E} = -\nabla\Phi = -\left[\overline{i_x}\frac{\partial\Phi}{\partial x} + \overline{i_y}\frac{\partial\Phi}{\partial y}\right]$$

$$\overline{E} = \begin{cases} \frac{V_0 k}{\sinh ka}\left[-\sin kx \sinh k(y-a)\overline{i_x} + \cos kx \cosh k(y-a)\overline{i_y}\right] & 0 < x < a \\ \frac{V_0 k}{\sinh kb}\left[\sin kx \sinh k(y+b)\overline{i_x} - \cos kx \cosh k(y+b)\overline{i_y}\right] & -b < x < 0 \end{cases}$$

C

Question: **What are the free surface charge distributions at $y = -b, y = 0$, and $y = a$?**

Solution:

$$\begin{aligned} \sigma_f(x, y=-b) &= \varepsilon_0 E_y(x, y=-b) = -\frac{\varepsilon_0 V_0 k}{\sinh kb}\cos kx \\ \sigma_f(x, y=a) &= -\varepsilon_0 E_y(x, y=a) = -\frac{\varepsilon_0 V_0 k}{\sinh ka}\cos kx \\ \sigma_f(x, y=0) &= \varepsilon_0\left[E_y(x, y=0_+) - E_y(x, y=0_-)\right] \\ &= \varepsilon_0\left[\frac{V_0 k}{\sinh ka}\cosh ka + \frac{V_0 k}{\sinh kb}\cosh kb\right]\cos kx \\ &= \varepsilon_0 V_0 k \cos kx\left[\coth ka + \coth kb\right] \end{aligned}$$

D

Question: What are the x and y components of force per unit area on a wavelength of width $\frac{2\pi}{k}$ on the $y = 0$ electric potential sheet?

Hint:

$$\frac{k}{2\pi}\int_{-\frac{\pi}{k}}^{+\frac{\pi}{k}} \cos^2 kx dx = \frac{1}{2}$$

$$\frac{k}{2\pi}\int_{-\frac{\pi}{k}}^{+\frac{\pi}{k}} \cos kx \sin kx dx = 0$$

Solution:

i)

$$\begin{aligned}
\left(\frac{\text{Force}}{\text{area}}\right)_y &= \frac{1}{2}\sigma_f(x, y=0)\left[E_y(x, y=0_+) + E_y(x, y=0_-)\right] \\
&= \frac{1}{2}\varepsilon_0\left[E_y(x, y=0_+) - E_y(x, y=0_-)\right]\left[E_y(x, y=0_+) + E_y(x, y=0_-)\right] \\
&= \frac{1}{2}\varepsilon_0\left[\left[E_y(x, y=0_+)\right]^2 - \left[E_y(x, y=0_-)\right]^2\right] \\
&= \frac{1}{2}\varepsilon_0\left(V_0 k\cos kx\right)^2\left[\coth^2 ka - \coth^2 kb\right] \\
\left(\frac{\text{Force}}{\text{area}}\right)_x &= \sigma_f(x, y=0)E_x(x, y=0_+) = \sigma_f(x, y=0)E_x(x, y=0_-) \\
&= \varepsilon_0 V_0 k\cos kx\left[\coth ka + \coth kb\right]V_0 k\sin kx \\
&= \varepsilon_0 V_0^2 k^2 \sin kx\cos kx\left[\coth ka + \coth kb\right] \\
\left\langle\frac{\text{Force}}{\text{area}}\right\rangle_y &= \frac{k}{2\pi}\int_{-\frac{\pi}{k}}^{+\frac{\pi}{k}}\left(\frac{\text{Force}}{\text{area}}\right)_y dx \\
&= \frac{k}{2\pi}\frac{1}{2}\varepsilon_0 V_0^2 k^2\left[\coth^2 ka - \coth^2 kb\right]\int_{-\frac{\pi}{k}}^{+\frac{\pi}{k}}\cos^2 kx dx \\
&= \frac{1}{4}\varepsilon_0 V_0^2 k^2\left[\coth^2 ka - \coth^2 kb\right] \\
\left\langle\frac{\text{Force}}{\text{area}}\right\rangle_x &= \frac{k}{2\pi}\int_{-\frac{\pi}{k}}^{+\frac{\pi}{k}}\left(\frac{\text{Force}}{\text{area}}\right)_x dx \\
&= \frac{k}{2\pi}\varepsilon_0 V_0^2 k^2\left[\coth ka + \coth kb\right]\int_{-\frac{\pi}{k}}^{+\frac{\pi}{k}}\sin kx\cosh kx dx \\
&= 0
\end{aligned}$$

(ii) Maxwell Stress Tensor Approach

$$\left(\frac{\text{Force}}{\text{area}}\right)_y = T_{yy}(x, y = 0_+) - T_{yy}(x, y = 0_-)$$
$$= \frac{1}{2}\varepsilon_0(E_y^2(x, y = 0_+) - E_x^2(x, y = 0_+)) - \frac{1}{2}\varepsilon_0(E_y^2(x, y = 0_-) - E_x^2(x, y = 0_-))$$
$$= \frac{1}{2}\varepsilon_0\left[E_y^2(x, y = 0_+) - E_y^2(x, y = 0_-)\right]$$
$$(E_x(x, y = 0_+) = E_x(x, y = 0_-))$$
$$\left(\frac{\text{Force}}{\text{Area}}\right)_x = T_{xy}(x, y = 0_+) - T_{xy}(x, y = 0_-)$$
$$= \varepsilon_0 E_x(x, y = 0_+)E_y(x, y = 0_+) - \varepsilon_0 E_x(x, y = 0_-)E_y(x, y = 0_-)$$
$$= \varepsilon_0 E_x(x, y = 0)\left[E_y(x, y = 0_+) - E_y(x, y = 0_-)\right]$$
$$= \sigma_f(x, y = 0)E_x(x, y = 0)$$

Problem 2

$$\chi(r, \theta) = \left(\frac{m_0}{4\pi r^2} + Ar\right)\cos\theta \qquad r < R$$

A

Question: Find the magnitude and direction of magnetic field in the region $0 < r < R$.

Solution:

$$\overline{H} = -\nabla\chi(r, \theta) = -\left[\frac{\partial\chi}{\partial r}\overline{i_r} + \frac{1}{r}\frac{\partial\chi}{\partial\theta}\overline{i_\theta}\right]$$
$$= -\left[\left(-\frac{2m_0}{4\pi r^3} + A\right)\cos\theta\overline{i_r} - \left(\frac{m_0}{4\pi r^3} + A\right)\sin\theta\overline{i_\theta}\right]$$
$$H_r(r = R, \theta) = 0 = -\left(A - \frac{2m_0}{4\pi R^3}\right)\cos\theta \Rightarrow A = \frac{2m_0}{4\pi R^3}$$
$$\overline{H} = -\frac{m_0}{4\pi R^3}\left[\left(1 - \frac{R^3}{r^3}\right)2\cos\theta\overline{i_r} - \left(\frac{R^3}{r^3} + 2\right)\sin\theta\overline{i_\theta}\right]$$

B

Question: What is the surface current on the $r = R$ surface?

Solution:

$$K_\phi(r = R, \theta) = H_\theta(r = R, \theta) = \frac{3m_0}{4\pi R^3}\sin\theta$$

C

Question: What is the equation of the magnetic field line that passes through the point $(r = r_0, \theta = \theta_0)$?

Hint:

$$\int \cot d\theta = \ln\left[\sin\theta\right]$$

$$\int \frac{2+\left(\frac{R}{r}\right)^3}{r\left(1-\left(\frac{R}{r}\right)^3\right)} dr = \ln\left[\frac{\left(\frac{r}{R}\right)^3 - 1}{\left(\frac{r}{R}\right)}\right]$$

<u>Solution:</u>

$$\frac{dr}{rd\theta} = \frac{H_r}{H_\theta} = \frac{2\cos\theta\left(1-\frac{R^3}{r^3}\right)}{-\left(\frac{R^3}{r^3}+2\right)\sin\theta}$$

$$\frac{\left(\frac{R^3}{r^3}+2\right)}{r\left(1-\frac{R^3}{r^3}\right)} dr = -2\cot\theta d\theta$$

$$\int \frac{\left(\frac{R^3}{r^3}+2\right)}{r\left(1-\frac{R^3}{r^3}\right)} dr = -\ln\left[\frac{r/R}{\left(\frac{r}{R}\right)^3 - 1}\right] + C_1$$

$$\int \cot\theta = \ln(\sin\theta) + C_2$$

$$-\ln\left[\frac{\frac{r}{R}}{\left(\frac{r}{R}\right)^3 - 1}\right] + 2\ln(\sin\theta) = \text{constant}$$

$$-\ln\left[\frac{\frac{r}{R}}{\left[\left(\frac{r}{R}\right)^3 - 1\right]\sin^2\theta}\right] = \text{constant}$$

$$\frac{\sin^2\theta}{\left(\frac{r}{R}\right)}\left[\left(\frac{r}{R}\right)^3 - 1\right] = \frac{\sin^2\theta_0\left(\left(\frac{r_0}{R}\right)^3 - 1\right)}{\left(\frac{r_0}{R}\right)}$$

Problem 3

A

<u>Question:</u> Find the magnetic field $\overline{H}$ in the gap free space region and in the nonlinear magnetic material.

<u>Solution:</u> In gap

$$H = \frac{Ni}{a}$$

(in free space and in nonlinear material)

B

Question: What is the total magnetic flux linked by the coil?

Solution:

$$\Phi = \mu_0 H(s-x)d + (\alpha H^3 + \mu H)xd$$
$$= \frac{Nid}{a}\left[(\mu-\mu_0)x + \mu_0 s + \alpha\left(\frac{Ni}{a}\right)^2 x\right]$$
$$\lambda = N\Phi = \frac{N^2 d}{a} i\left[(\mu-\mu_0)x + \mu_0 s + \alpha\left(\frac{Ni}{a}\right)^2 x\right]$$

C

Question: What is the voltage across the coil terminals when the nonlinear material is stationary, that is, x is constant and $i = I_0 \sin\omega t$?

Solution: With x = constant

$$v = \frac{d\lambda}{dt} = \frac{N^2 d}{a}\left[[(\mu-\mu_0)x + \mu_0 s]\frac{di}{dt} + \frac{\alpha N^2}{a^2} x 3i^2 \frac{di}{dt}\right]$$
$$= \frac{N^2 d}{a}\frac{di}{dt}\left[[(\mu-\mu_0)x + \mu_0 s] + \frac{3\alpha N^2}{a^2} x i^2\right]$$

$$i(t) = I_0 \sin\omega t$$
$$v(t) = \frac{N^2 d}{a} I_0 \omega \cos\omega t\left[[(\mu-\mu_0)x + \mu_0 s] + \frac{3\alpha N^2 x I_0^2}{a^2}\sin^2\omega t\right]$$

D

Question: What is the force on the nonlinear magnetic material for any i?

Solution:

$$dw'_m(i,x) = \lambda di + f_x dx$$
$$w'_m(i,x) = \int_{x=\text{constant}} \lambda di = \frac{N^2 d}{a}\left[[(\mu-\mu_0)x + \mu_0 s]\frac{i^2}{2} + \frac{\alpha N^2}{a^2}\frac{i^4}{4} x\right]$$
$$f_x = \left.\frac{\partial w'_m(i,x)}{\partial x}\right|_i = \frac{N^2 d}{a}\left[(\mu-\mu_0)\frac{i^2}{2} + \frac{\alpha N^2}{a^2}\frac{i^4}{4}\right]$$

Problem 4

A

Question: Solve for E_1 and E_2 as a function of $V_0, a, b, \varepsilon, \varepsilon_0$ and $\xi(x,t)$.

Solution:

$$E_1(b-\xi)+E_2 a = V_0$$

$$\varepsilon_0 E_1 = \varepsilon E_2 \Rightarrow E_2 = \frac{\varepsilon_0}{\varepsilon} E_1$$

$$E_1\left[b-\xi+\frac{\varepsilon_0 a}{\varepsilon}\right] = V_0$$

$$E_1 = \frac{V_0}{b-\xi+\frac{\varepsilon_0 a}{\varepsilon}} = \frac{V_0}{\left(b+\frac{\varepsilon_0 a}{\varepsilon}\right)\left[1-\frac{\xi}{\left(b+\frac{\varepsilon_0 a}{\varepsilon}\right)}\right]}$$

B

Question: Linearize E_1 for small deflections in $\xi(x,t)$

Solution:

$$E_1 \approx \frac{V_0\left[1+\frac{\xi}{\left(b+\frac{\varepsilon_0 a}{\varepsilon}\right)}\right]}{\left(b+\frac{\varepsilon_0 a}{\varepsilon}\right)}$$

C

Question: Find the electrical force per unit area on the membrane and linearize for small deflections in $\xi(x,t)$.

Solution:

$$\left(\frac{\text{Force}}{\text{Area}}\right)_y = T_{yy} = \frac{1}{2}\varepsilon_0 E_1^2$$

$$= \frac{1}{2}\frac{\varepsilon_0 V_0^2}{\left(b+\frac{\varepsilon_0 a}{\varepsilon}\right)^2}\left[1+\frac{2\xi}{\left(b+\frac{\varepsilon_0 a}{\varepsilon}\right)}\right]$$

D

Question: Write the linearized governing force balance equation for the membrane to first order in displacement $\xi(x,t)$ including the inertial, membrane tension, gravity, and electrical forces.

Solution:

$$\sigma_m \frac{\partial^2 \xi}{\partial t^2} = S\frac{\partial^2 \xi}{\partial x^2} - \sigma_m g + \frac{1}{2}\frac{\varepsilon_0 V_0^2}{\left(b+\frac{\varepsilon_0 a}{\varepsilon}\right)^2}\left[1+\frac{2\xi}{\left(b+\frac{\varepsilon_0 a}{\varepsilon}\right)}\right]$$

E

Question: What voltage V_0 is required to remove membrane sag in equilibrium so that $\xi(x,t) = 0$?

Solution:

$$\xi(x,t) = 0 \Rightarrow \sigma_m g = \frac{1}{2}\frac{\varepsilon_0 V_0^2}{\left(b + \frac{\varepsilon_0 a}{\varepsilon}\right)^2}$$

$$V_0 = \left[\frac{2\sigma_m g}{\varepsilon_0}\right]^{\frac{1}{2}}\left(b + \frac{\varepsilon_0 a}{\varepsilon}\right)$$

F

Question: For perturbation deflections around the $\xi(x,t) = 0$ equilibrium of the form $\xi(x,t) = \mathrm{Re}\left[\hat{\xi}e^{j(\omega t - kx)}\right]$, what is the $\omega - k$ dispersion relation?

Solution:

$$\xi(x,t) = \mathrm{Re}\left[\hat{\xi}e^{j(\omega t - kx)}\right]$$

$$-\sigma_m\omega^2 = -Sk^2 + \frac{\varepsilon_0 V_0^2}{\left(b + \frac{\varepsilon_0 a}{\varepsilon}\right)^3}$$

$$\omega^2 = k^2 v_p^2 - \frac{\varepsilon_0 V_0^2}{\sigma_m\left(b + \frac{\varepsilon_0 a}{\varepsilon}\right)^3}, \qquad v_p^2 = \frac{S}{\sigma_m}$$

G

Question: The membrane ends are fixed to the supports at $x = 0$ and $x = l$ so that $\xi(x = 0, t) = \xi(x = l, t) = 0$. For a given real value of ω what are the allowed values of k that satisfy the boundary conditions?

Solution:

$$k^2 = \frac{\omega^2}{v_p^2} + \frac{\varepsilon_0 V_0^2}{\sigma_m\left(b + \frac{\varepsilon_0 a}{\varepsilon}\right)^3}$$

$$k = \pm k_0 = \pm\left[\frac{\omega^2}{v_p^2} + \frac{\varepsilon_0 V_0^2}{\sigma_m\left(b + \frac{\varepsilon_0 a}{\varepsilon}\right)^3}\right]^{\frac{1}{2}}$$

$$\xi(x,t) = \mathrm{Re}\left[\left(\hat{\xi}_1 e^{jk_0 x} + \hat{\xi}_2 e^{-jk_0 x}\right)e^{j\omega t}\right]$$

$$\hat{\xi}(x = 0, t) = 0 = \hat{\xi}_1 + \hat{\xi}_2 \Rightarrow \hat{\xi}_1 = -\hat{\xi}_2$$

$$\begin{aligned}\hat{\xi}(x = l, t) = 0 &= \hat{\xi}_1 e^{jk_0 l} + \hat{\xi}_2 e^{-jk_0 l} \\ &= \hat{\xi}_1\left(e^{jk_o l} - e^{-jk_0 l}\right) \\ &= 2j\hat{\xi}_1 \sin k_0 l\end{aligned}$$

$$\sin k_0 l = 0 \Rightarrow k_0 l = n\pi, n = 1, 2, \ldots$$
$$k_0 = \frac{n\pi}{l}$$

H

Question: Under what conditions is the equilibrium of part (e) stable?
Solution:

$$\omega^2 = v_p^2 \left[\left(\frac{n\pi}{l}\right)^2 - \frac{\varepsilon_0 V_0^2}{v_p^2 \sigma_m \left(b + \frac{\varepsilon_0 a}{\varepsilon}\right)^3} \right]$$

For stability,

$$\omega^2 > 0 \Rightarrow \left(\frac{\pi}{l}\right)^2 > \frac{\varepsilon_0 V_0^2}{\sigma_m \left(b + \frac{\varepsilon_0 a}{\varepsilon}\right)^3 v_p^2}$$

$$\left(\frac{\pi}{l}\right)^2 > \frac{\varepsilon_0 V_0^2}{S \left(b + \frac{\varepsilon_0 a}{\varepsilon}\right)^3}$$

I

Question: What is the maximum membrane mass density per unit area, σ_m, that the voltage V_0 can stably remove sag so that $\xi(x,t) = 0$?

Solution: From part (e):

$$\frac{\varepsilon_0 V_0^2}{\left(b + \frac{\varepsilon_0 a}{\varepsilon}\right)^2} = 2\sigma_m g$$

$$\left(\frac{\pi}{l}\right)^2 > \frac{2\sigma_m g}{S \left(b + \frac{\varepsilon_0 a}{\varepsilon}\right)} \Rightarrow \sigma_m < \frac{S \left(\frac{\pi}{l}\right)^2 \left(b + \frac{\varepsilon_0 a}{\varepsilon}\right)}{2g}$$