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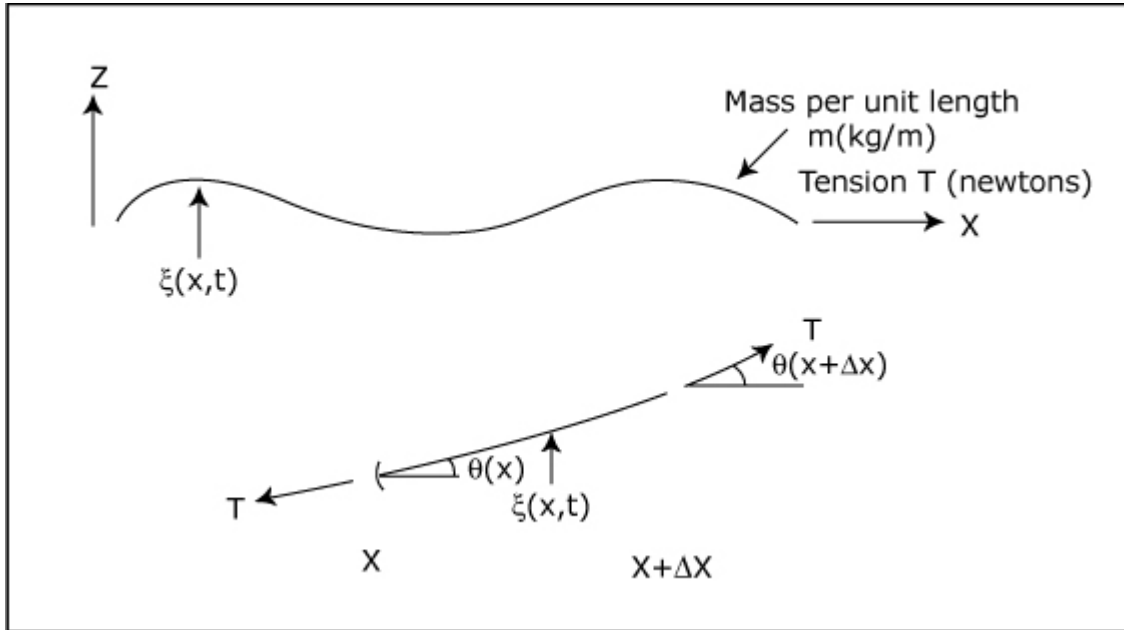
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6.641, Electromagnetic Fields, Forces, and Motion  
 Prof. Markus Zahn  
**Lecture 18: Waves and Instabilities In Elastic Media**

I. Transverse Motions of Wires Under Tension



$$m \Delta x \frac{\partial^2 \xi}{\partial t^2} = \frac{T_z(x+\Delta x) - T_z(x)}{\Delta x} + F_{\text{ext}} \Delta x$$

Force per unit length (nt/m)

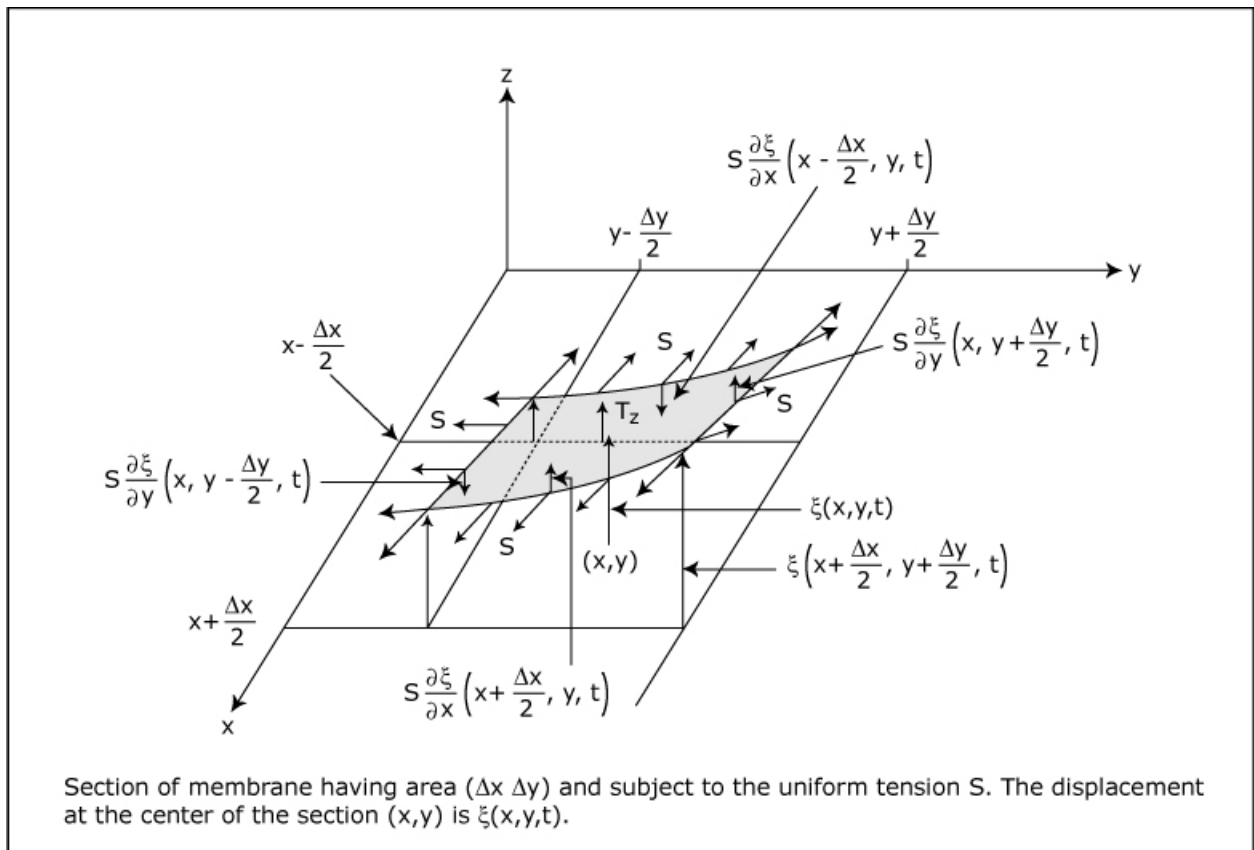
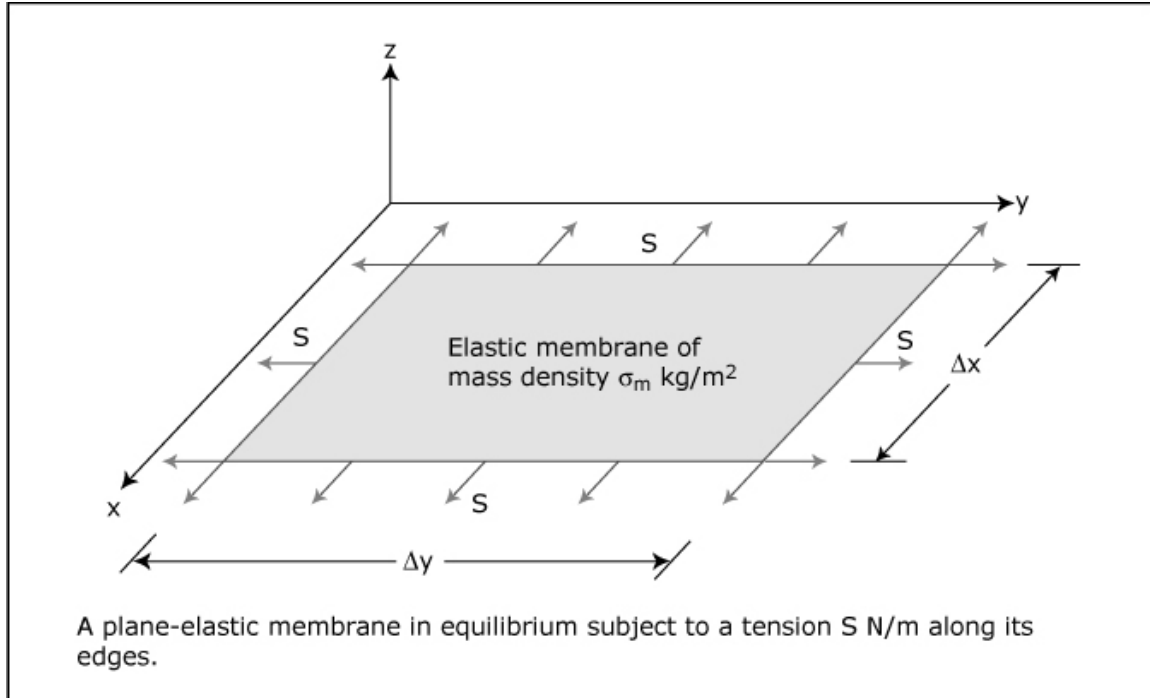
$$m \frac{\partial^2 \xi}{\partial t^2} = \frac{\partial T_z}{\partial x} + F_{\text{ext}}$$

$$T_z = T \sin \theta \approx T \tan \theta \approx T \left[ \frac{\xi(x+\Delta x) - \xi(x)}{\Delta x} \right]$$

$$\approx T \frac{\partial \xi}{\partial x}$$

$$m \frac{\partial^2 \xi}{\partial t^2} = T \frac{\partial^2 \xi}{\partial x^2} + F_{\text{ext}}$$

## II. Transverse Motions of Membranes



$$\sigma_m \Delta x \Delta y \frac{\partial^2 \xi}{\partial t^2} = S \Delta x \left[ \frac{\partial \xi}{\partial y} \left( x, y + \frac{\Delta y}{2}, t \right) - \frac{\partial \xi}{\partial y} \left( x, y - \frac{\Delta y}{2}, t \right) \right]$$

↑  
membrane mass per  
unit area (kg/m<sup>2</sup>)

$$+ S \Delta y \left[ \frac{\partial \xi}{\partial x} \left( x + \frac{\Delta x}{2}, y, t \right) - \frac{\partial \xi}{\partial x} \left( x - \frac{\Delta x}{2}, y, t \right) \right]$$

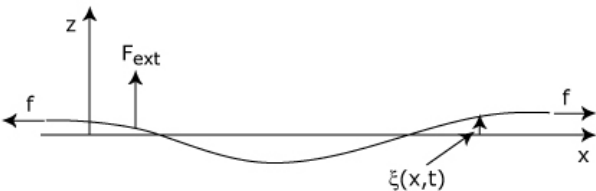
↑  
membrane tension (newtons/m)

$$+ T_z \Delta x \Delta y$$

↑  
external force per unit area (newtons/m<sup>2</sup>)

$$\sigma_m \frac{\partial^2 \xi}{\partial t^2} = S \left( \frac{\partial^2 \xi}{\partial x^2} + \frac{\partial^2 \xi}{\partial y^2} \right) + T_z$$

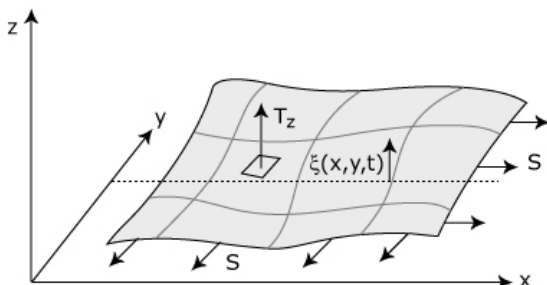
**WIRE OR "STRING"**



$m \frac{\partial^2 \xi}{\partial t^2} = T \frac{\partial^2 \xi}{\partial x^2} + F_{\text{ext}}$

ξ – transverse displacement  
m – mass/unit length  
T – tension (constant force)  
F<sub>ext</sub> – transverse force/unit length

**MEMBRANE**



$\sigma_m \frac{\partial^2 \xi}{\partial t^2} = S \left( \frac{\partial^2 \xi}{\partial x^2} + \frac{\partial^2 \xi}{\partial y^2} \right) + T_z$

ξ – transverse displacement  
σ<sub>m</sub> – surface mass density  
S – tension in y- and z- directions  
(constant force per unit length)  
T<sub>z</sub> – z-directed force per unit area

### III. Non-Dispersive Waves on a String ( $F_{\text{ext}} = 0$ )

#### A. Dispersion Equation

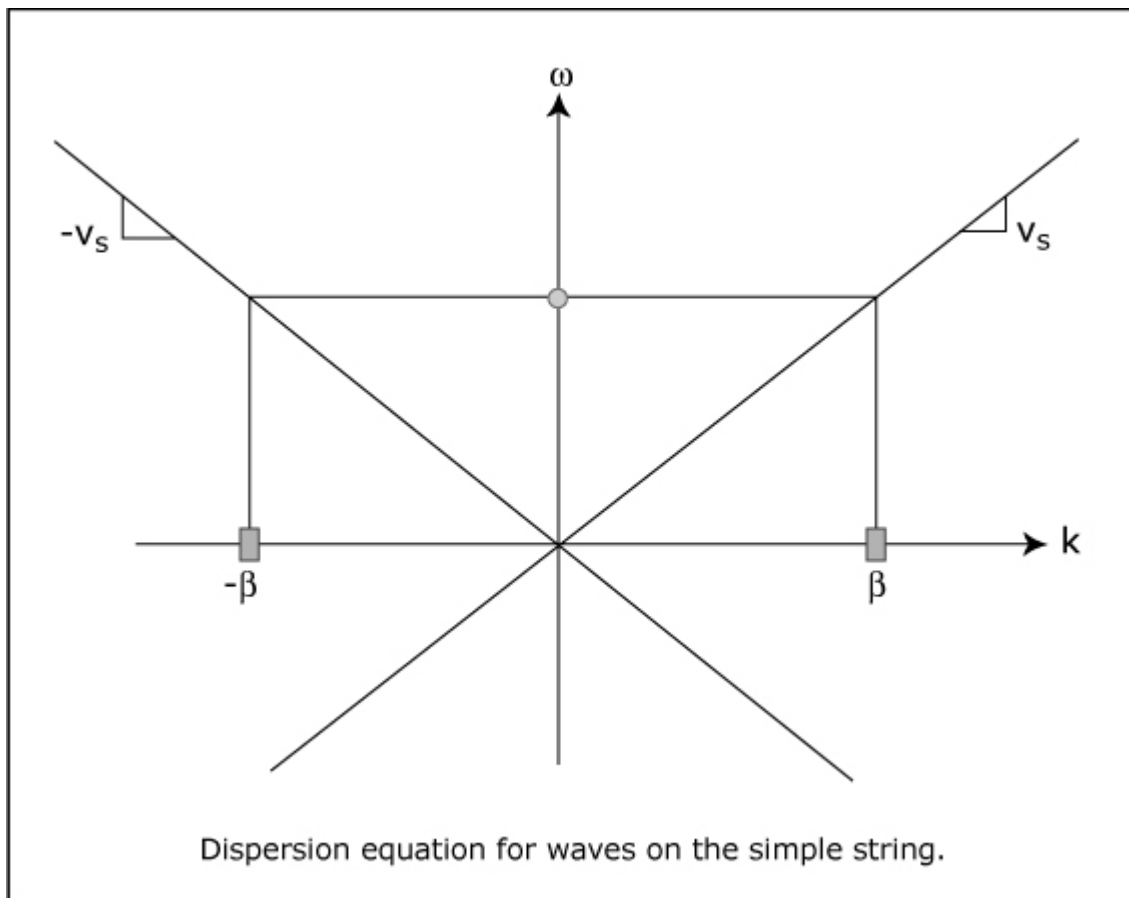
$$\mu \frac{\partial^2 \xi}{\partial t^2} = \frac{T}{m} \frac{\partial^2 \xi}{\partial x^2}$$

$$v_s = \sqrt{T/m}$$

$$\frac{\partial^2 \xi}{\partial t^2} = v_s^2 \frac{\partial^2 \xi}{\partial x^2} \quad (\text{wave equation})$$

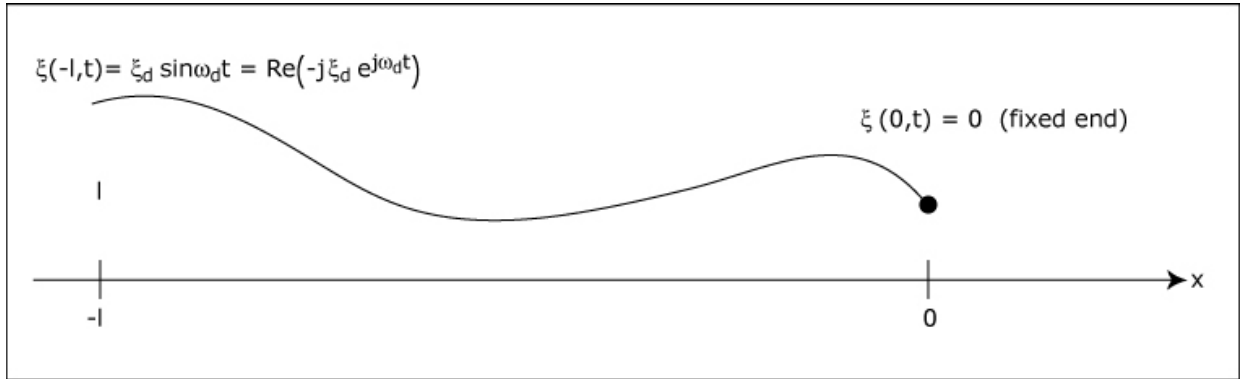
$$\xi = \text{Re} \left[ \hat{\xi} e^{j(\omega t - kx)} \right]$$

$$-\omega^2 \hat{\xi} = -k^2 v_s^2 \hat{\xi} \Rightarrow \omega^2 = k^2 v_s^2 \Rightarrow \omega = \pm k v_s$$



## B. Driven and Transient Responses

$$\xi(-l, t) = \xi_d \sin \omega_d t = \operatorname{Re}(-j \xi_d e^{j \omega_d t})$$



$$k = \pm \frac{\omega_d}{v_s}$$

$$\xi(x, t) = \operatorname{Re} \left[ \left( \hat{\xi}_1 e^{-\frac{j \omega_d x}{v_s}} + \hat{\xi}_2 e^{+\frac{j \omega_d x}{v_s}} \right) e^{j \omega_d t} \right]$$

$$\xi(0, t) = 0 = \operatorname{Re} \left[ \left( \hat{\xi}_1 + \hat{\xi}_2 \right) e^{j \omega_d t} \right] \Rightarrow \hat{\xi}_1 = -\hat{\xi}_2$$

$$\begin{aligned} \hat{\xi}(-l, t) &= -j \xi_d = \hat{\xi}_1 e^{\frac{j \omega_d l}{v_s}} + \hat{\xi}_2 e^{-\frac{j \omega_d l}{v_s}} = \hat{\xi}_1 \left( e^{\frac{j \omega_d l}{v_s}} - e^{-j \omega_d l / v_s} \right) \\ &= 2j \hat{\xi}_1 \sin \frac{\omega_d l}{v_s} \end{aligned}$$

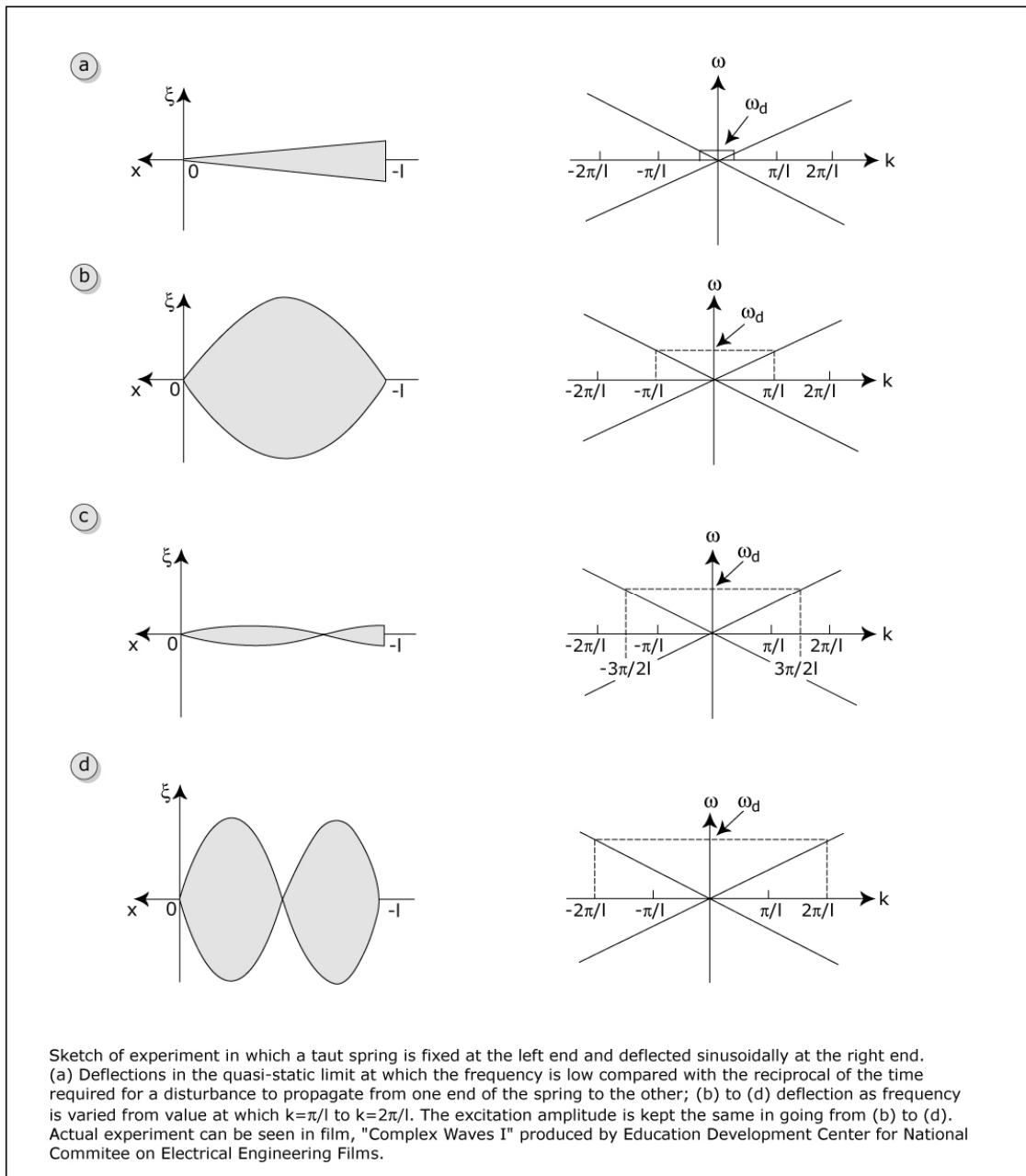
$$-j \xi_d = 2j \sin \frac{\omega_d l}{v_s} \hat{\xi}_1 \Rightarrow \hat{\xi}_1 = -\frac{\xi_d}{2 \sin \frac{\omega_d l}{v_s}}$$

$$\begin{aligned} \hat{\xi}(x) &= -\frac{\xi_d}{2 \sin \frac{\omega_d l}{v_s}} \left( e^{-\frac{j \omega_d x}{v_s}} - e^{\frac{j \omega_d x}{v_s}} \right) = \frac{+\xi_d}{2 \sin \frac{\omega_d l}{v_s}} \left( +j \sin \frac{\omega_d x}{v_s} \right) \\ &= \frac{j \sin \frac{\omega_d x}{v_s} \xi_d}{\sin \frac{\omega_d l}{v_s}} \end{aligned}$$

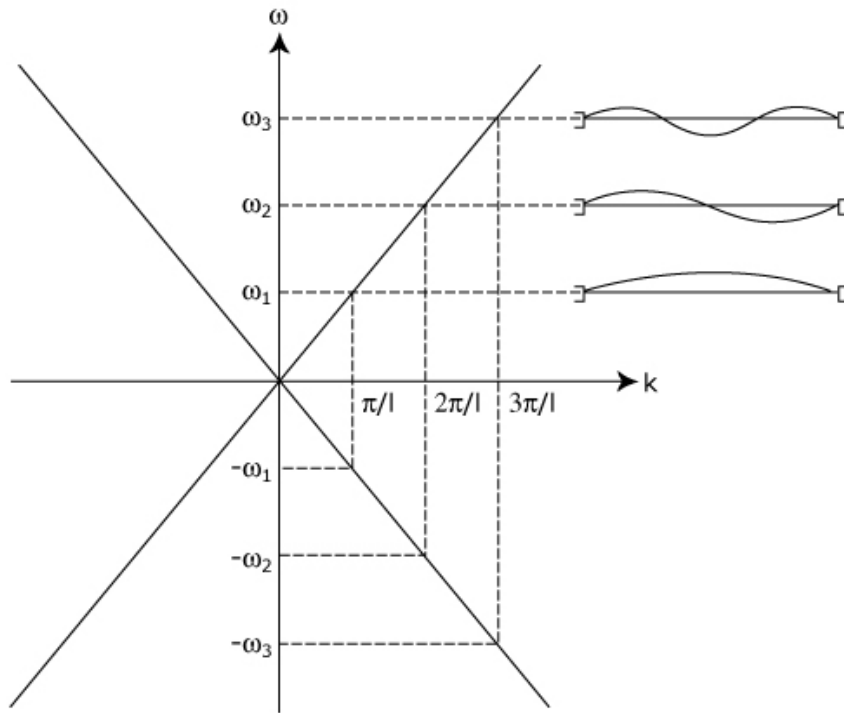
$$\xi(x, t) = \text{Re} \left[ \hat{\xi}(x) e^{j\omega_d t} \right] = \text{Re} \left[ j \frac{\sin \frac{\omega_d x}{v_s}}{\sin \frac{\omega_d l}{v_s}} \xi_d e^{j\omega_d t} \right] = - \frac{\xi_d \sin \frac{\omega_d x}{v_s}}{\sin \frac{\omega_d l}{v_s}} \sin \omega_d t$$

$$\text{Resonance: } \sin \frac{\omega_d l}{v_s} = 0$$

$$\frac{\omega_d l}{v_s} = n\pi, \quad n=1, 2, 3, \dots$$



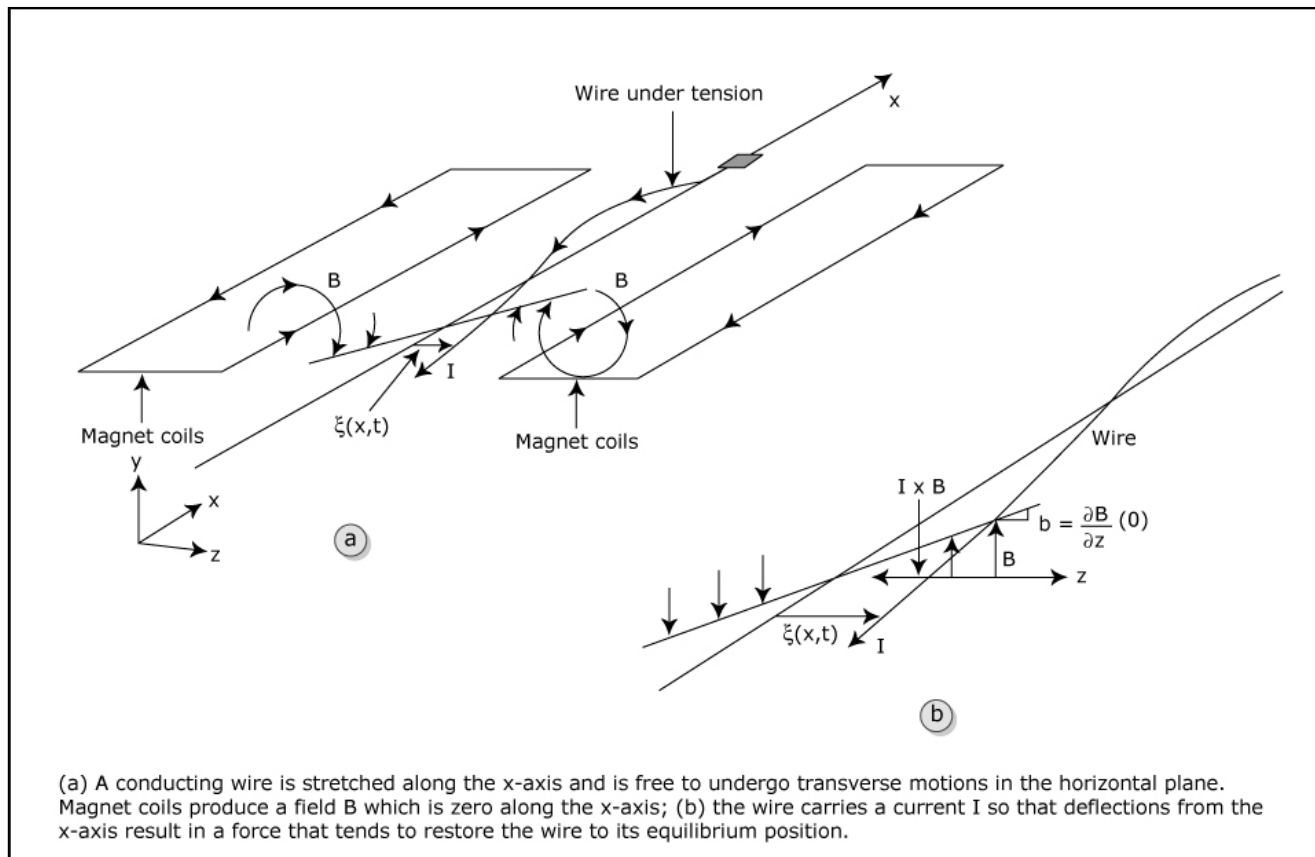
### Simple Elastic Continua



Allowed wavenumbers (eigenvalues)  $k = k_n$  as they are related to the eigenfrequencies  $\omega_n$  by the dispersion equation.



#### IV. Cut-off or Evanescent Waves



$$m \frac{\partial^2 \xi}{\partial t^2} = T \frac{\partial^2 \xi}{\partial x^2} + F_{\text{ext}}$$

$$F_{\text{ext}} = (\bar{I} \times \bar{B}) \cdot \bar{i}_z = -I \bar{i}_x \times \left( \frac{\partial B_y}{\partial z} \xi \bar{i}_y \right) \cdot \bar{i}_z$$

$$= -Ib \xi$$

$$m \frac{\partial^2 \xi}{\partial t^2} = \frac{T}{m} \frac{\partial^2 \xi}{\partial x^2} - \frac{Ib \xi}{m}$$

$$\frac{\partial^2 \xi}{\partial t^2} = v_s^2 \frac{\partial^2 \xi}{\partial x^2} - \omega_c^2 \xi ; \quad v_s^2 = \frac{T}{m}, \quad \omega_c^2 = \frac{Ib}{m}$$

$$\xi = \text{Re} \left[ \hat{\xi} e^{j(\omega t - kx)} \right]$$

$$+\omega^2 = +k^2 v_s^2 + \omega_c^2 \Rightarrow k = \pm \sqrt{\omega^2 - \omega_c^2} / v_s$$

$$\xi(0, t) = 0, \quad \xi(-l, t) = \xi_d \sin \omega_d t$$

$$k = \pm |k_r|, \quad \omega_d > \omega_c \quad \left( |k_r| = \sqrt{\omega_d^2 - \omega_c^2} / v_s \right)$$

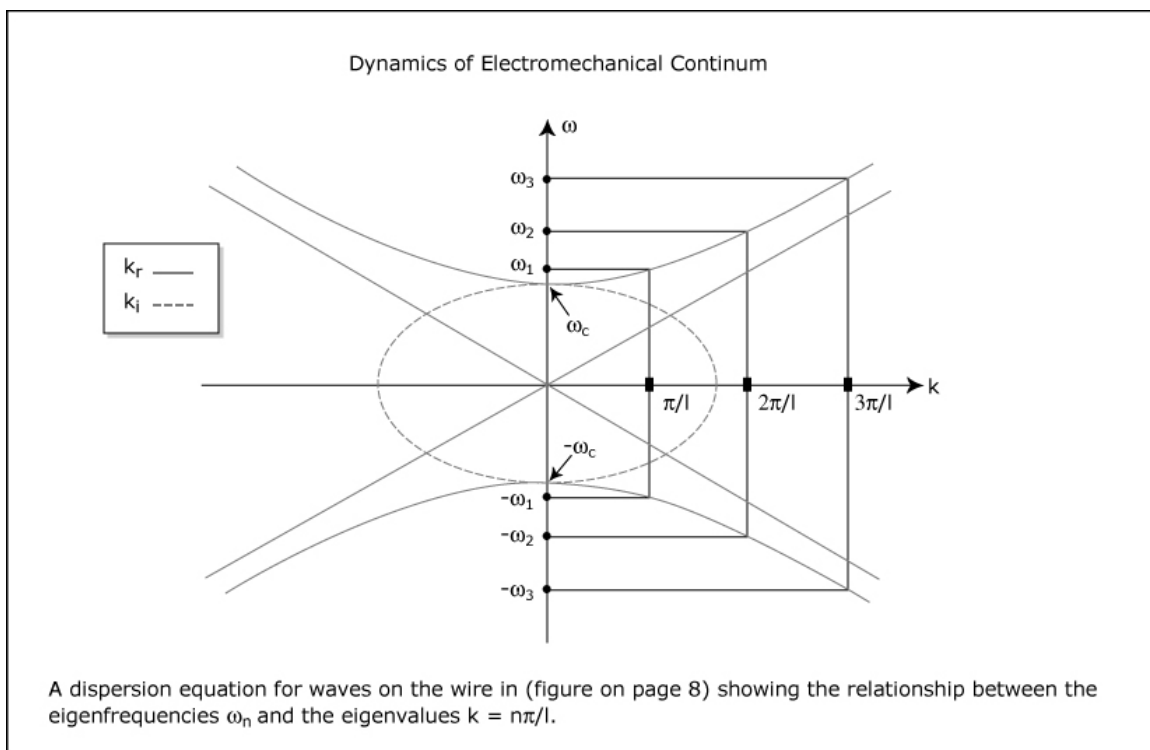
$$k = \pm j |k_i|, \quad \omega_d < \omega_c \quad \left( |k_i| = \sqrt{\omega_c^2 - \omega_d^2} / v_s \right)$$

$$\omega_d > \omega_c$$

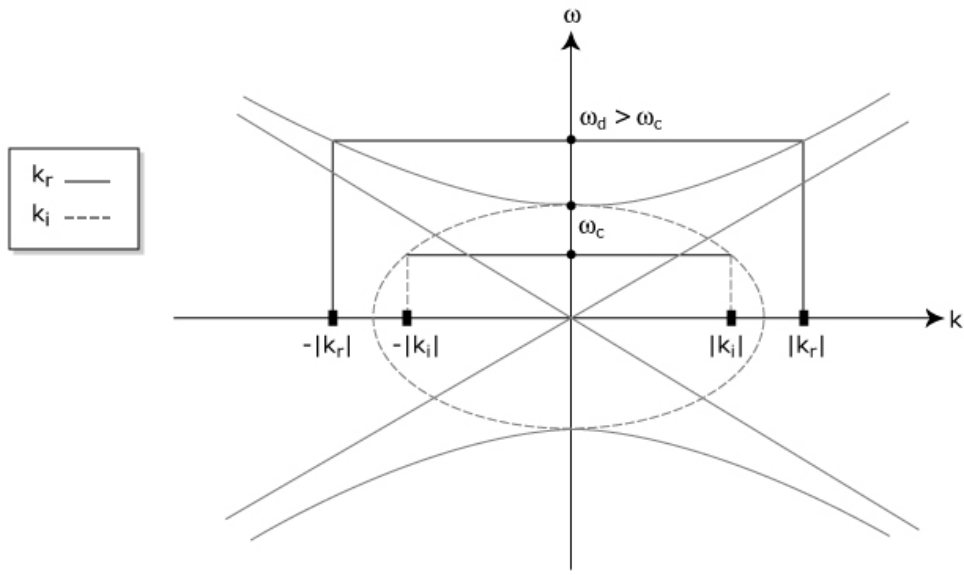
$$\xi = -\xi_d \frac{\sin |k_r| x}{\sin |k_r| l} \sin \omega_d t$$

$$\omega_d < \omega_c$$

$$\xi = -\xi_d \frac{\sinh |k_i| x}{\sinh |k_i| l} \sin \omega_d t$$



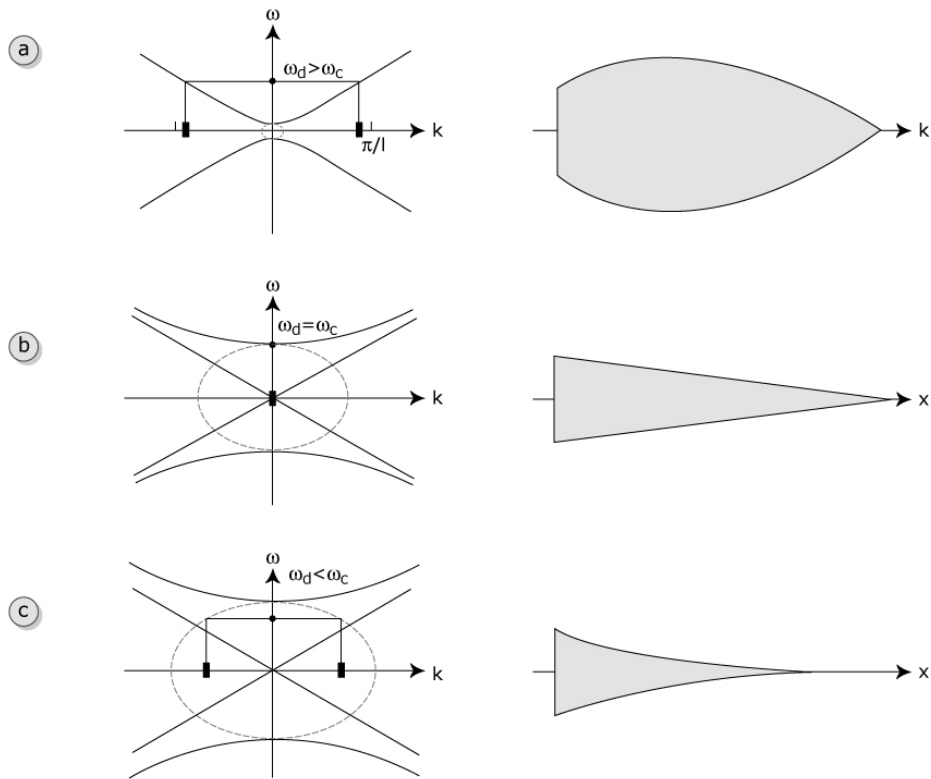
### Waves and Instabilities in Stationary Media



Dispersion relation for the wire subject to a restoring force distributed along its length (for the case shown in figure on page 8). Complex values of  $k$  are shown as functions of real values of  $\omega$ .

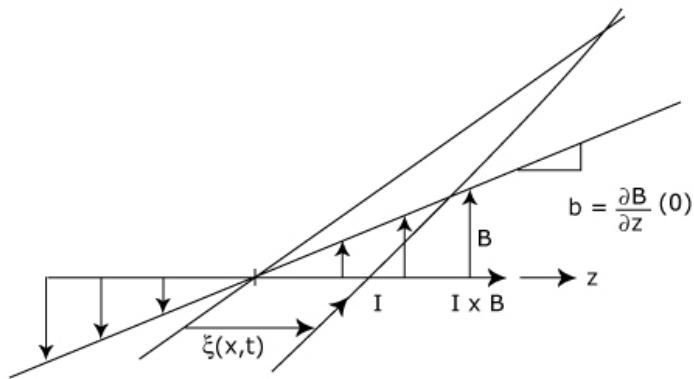
$$\text{resonance } (|k_r|l = n\pi) \Rightarrow \frac{\omega^2 - \omega_c^2}{v_s^2} = \left(\frac{n\pi}{l}\right)^2 \Rightarrow \omega = \left[ \omega_c^2 + \left(\frac{n\pi}{l}\right)^2 v_s^2 \right]^{1/2}$$

### Waves and Instabilities in Stationary Media



Envelope of wire deflection in magnetic field. The wire is fixed at the right end and driven at a fixed sinusoidal frequency at the left end. The  $\omega$ - $k$  plots show the effect of the current  $I$  on the dispersion equation. The current  $I$  (or cutoff frequency  $\omega_c$ ) is being raised so that (a),  $I \approx 0$ , (b)  $I$  is sufficient just to cut off the propagation ( $\omega_d = \omega_c$ ), and (c) the waves are evanescent,  $\omega_d < \omega_c$ . This experiment can be seen in the film "Complex Waves I," produced for the National Committee on Electrical Engineering films by Education Development Center, Newton, Mass.

### V. Absolute or Non convective Instability



Wire carrying current  $I$  in a magnetic field that is zero along the axis  $\xi=0$ . Current is reversed from the situation shown in figure on page 8.

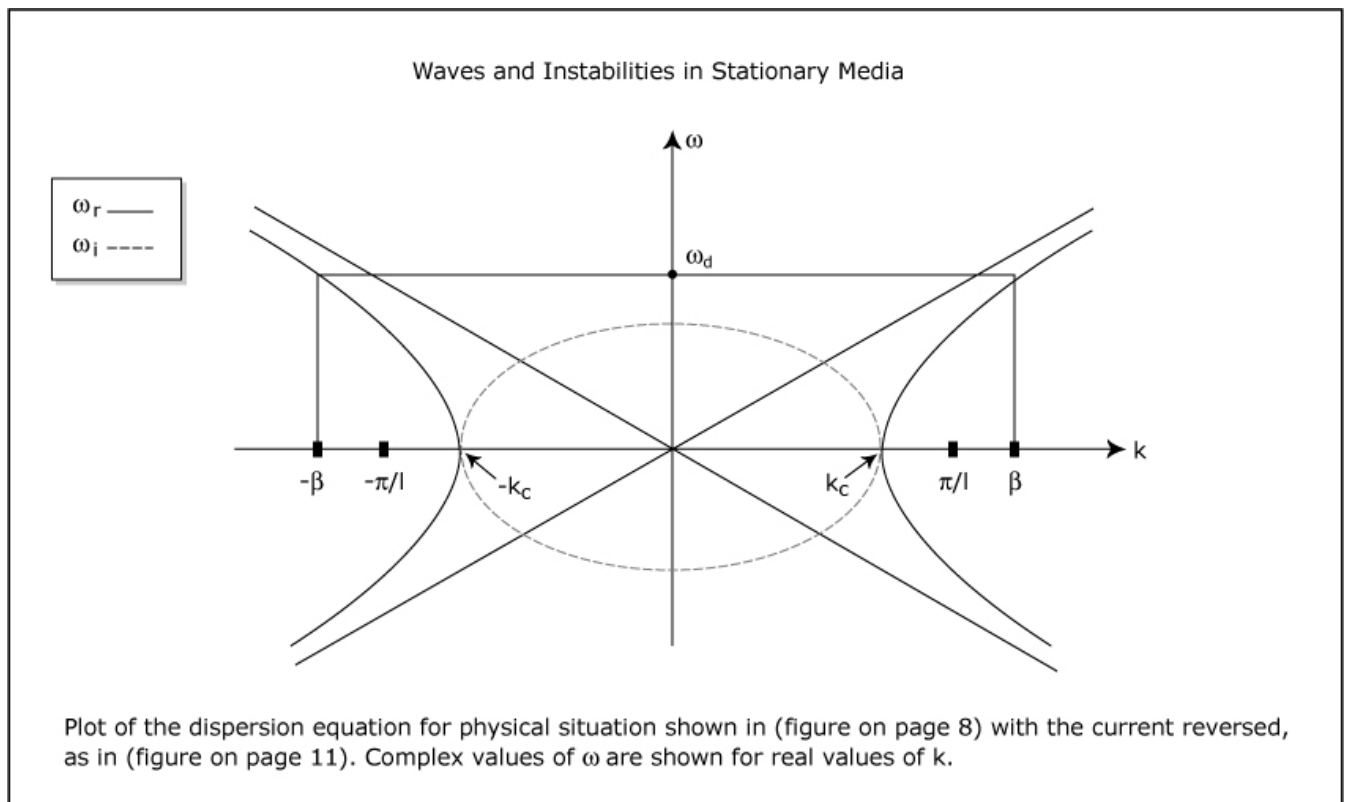
$$F_{\text{ext}} = +Ib\xi$$

$$m \frac{\partial^2 \xi}{\partial t^2} = \frac{T}{m} \frac{\partial^2 \xi}{\partial x^2} + \frac{Ib\xi}{m}$$

$$\frac{\partial^2 \xi}{\partial t^2} = v_s^2 \frac{\partial^2 \xi}{\partial x^2} + \omega_c^2 \xi, \quad v_s^2 = \frac{T}{m}, \quad \omega_c^2 = \frac{Ib}{m}$$

$$\xi = \text{Re} \left[ \hat{\xi} e^{j(\omega t - kx)} \right]$$

$$-\omega^2 = -k^2 v_s^2 + \omega_c^2 \Rightarrow \omega^2 = k^2 v_s^2 - \omega_c^2$$



Take undriven spring=  $\xi(-l, t) = 0, \quad \xi(0, t) = 0$

$$\xi = \text{Re} \left[ \hat{\xi}(x) e^{j\omega t} \right]$$

$$\hat{\xi}(x) = \xi_1 e^{-jkx} + \xi_2 e^{+jkx}$$

$$K = \sqrt{\frac{\omega^2 + \omega_c^2}{v_s^2}}$$

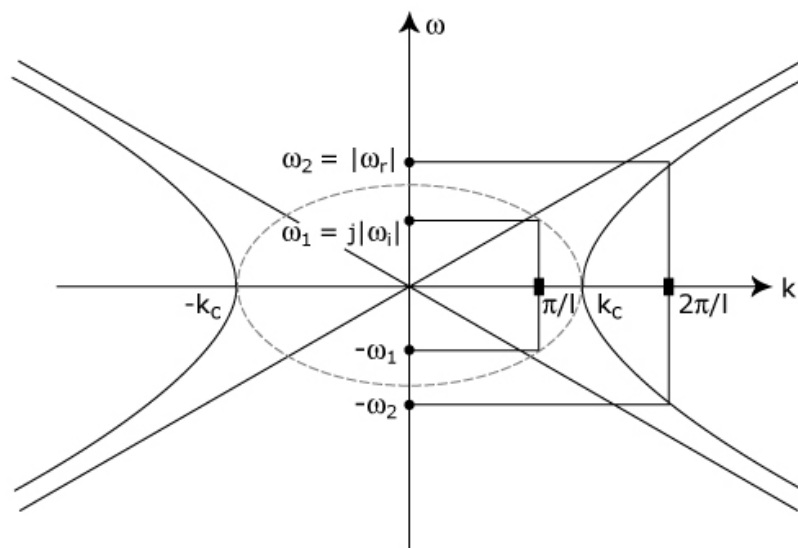
$$\hat{\xi}(x=0) = 0 = \xi_1 + \xi_2$$

$$\begin{aligned} \hat{\xi}(x=-l) = 0 &= \xi_1 e^{jkl} + \xi_2 e^{-jkl} = \xi_1 (e^{jkl} - e^{-jkl}) \\ &= 2j\xi_1 \operatorname{sinkl} \end{aligned}$$

$$kl = \frac{n\pi}{l}$$

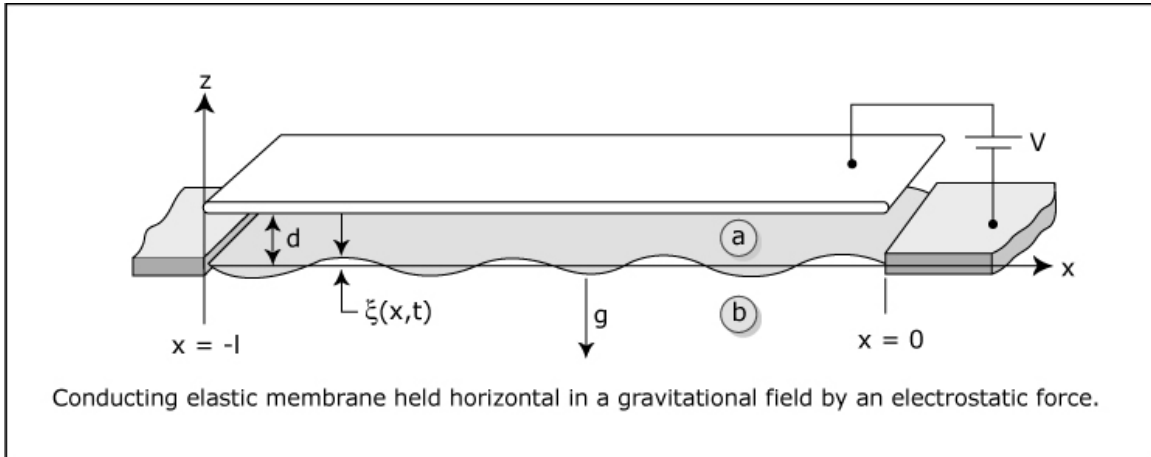
$$\omega^2 = \left(\frac{n\pi}{l}\right)^2 - \omega_c^2, \quad \text{if } \left(\frac{n\pi}{l}\right) < \omega_c, \quad \omega = \pm j|\omega_i|$$

Negative imaginary roots are absolutely unstable:  $\sim e^{|\omega_i|t}$



The dispersion equation for the system of (figure on page 8) with current as shown in figure on page 11. Complex values of  $\omega$  are shown for real values of  $k$ . The allowed values of  $k$  give rise to the eigenfrequencies as shown.

## VI. Electric Field Levitation of Membrane



$$\sigma_m \frac{\partial^2 \xi}{\partial t^2} = S \frac{\partial^2 \xi}{\partial x^2} - \sigma_m g + T_z^e$$

$$\vec{E} = \frac{-V}{d - \xi} \vec{i}_z$$

$$T_z^e = (T_{zj}^a - T_{zj}^b) n_j = T_{zz}^a = \frac{1}{2} \epsilon_0 E_z^2 = \frac{1}{2} \epsilon_0 \left( \frac{V}{d - \xi} \right)^2$$

$$= \frac{1}{2} \frac{\epsilon_0 V^2}{d^2} \left( \frac{1}{1 - (\xi/d)} \right)^2$$

$$\approx \frac{1}{2} \frac{\epsilon_0 V^2}{d^2} \left( 1 + 2 \frac{\xi}{d} \right)$$

$$\sigma_m \frac{\partial^2 \xi}{\partial t^2} = S \frac{\partial^2 \xi}{\partial x^2} - \sigma_m g + \frac{1}{2} \frac{\epsilon_0 V^2}{d^2} + \frac{\epsilon_0 V^2}{d^3} \xi$$

$$\text{Equilibrium: } \xi = 0 \Rightarrow \sigma_m g = \frac{1}{2} \frac{\epsilon_0 V^2}{d^2}$$

Perturbations :  ~~$\sigma_m$~~   $\frac{\partial^2 \xi}{\partial t^2} = \frac{S}{\sigma_m} \frac{\partial^2 \xi}{\partial x^2} + \frac{\epsilon_0 V^2}{d^3 \sigma_m} \xi$

$$v_s^2 = \frac{S}{\sigma_m}, \quad \frac{\epsilon_0 V^2}{d^3 \sigma_m} = \omega_c^2$$

$$\xi = \text{Re} \left[ \hat{\xi} e^{j(\omega t - kx)} \right]$$

$$-\omega^2 = -k^2 v_s^2 + \omega_c^2$$

$$\xi(0, t) = \xi(-l, t) = 0 \Rightarrow k = \frac{n\pi}{l}$$

$$\omega^2 = \left( \frac{n\pi}{l} v_s \right)^2 - \omega_c^2$$

$$\text{Stable if: } \omega_c^2 = \frac{\epsilon_0 V^2}{\sigma_m d^3} < \left( \frac{n\pi}{l} v_s \right)^2 \Rightarrow \frac{\epsilon_0 V^2}{\sigma_m d^3 v_s^2} = \frac{\epsilon_0 V^2}{d^3 S} < \left( \frac{n\pi}{l} \right)^2$$

First Unstable mode:  $n = 1$

$$\underbrace{\frac{\epsilon_0 V^2}{d^3 S}} < \left( \frac{\pi}{l} \right)^2$$

$$\frac{2\sigma_m g}{dS} < \left( \frac{\pi}{l} \right)^2 \Rightarrow \sigma_m < \frac{\left( \frac{\pi}{l} \right)^2 S d}{2g}$$