**Summary of binary detection with vector observation in iid Gaussian noise.:**

**First remove center point from signal and its effect on observation.**

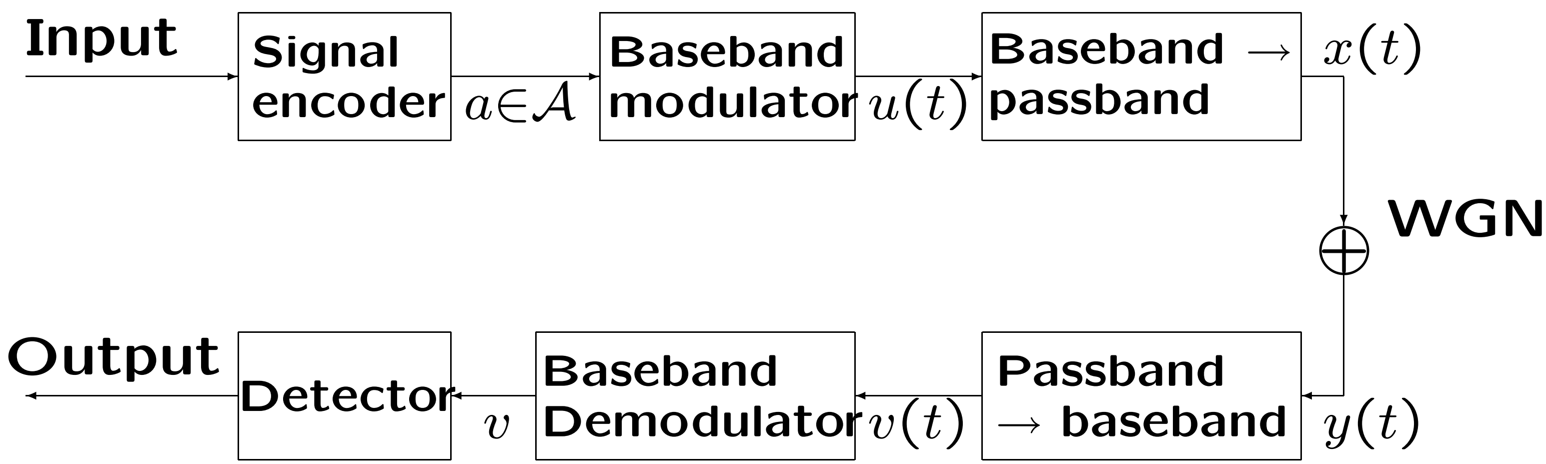
**Then signal is $\pm\vec{a}$. and $\vec{v} = \pm\vec{a} + \vec{Z}$.**

**Find $\langle\vec{v}, \vec{a}\rangle$ and compare with threshold (0 for ML case).**

**This does not depend on the vector basis - becomes trivial if $\vec{a}$ normalized is a basis vector.**

**Received components orthogonal to signal are irrelevant.**

# QAM DETECTION




We have seen how to design a MAP or ML detector from observing $v$. We generalize to an arbitrary complex signal set $\mathcal{A}$

We also question what the entire receiver should do from observation of $y(t)$ or $v(t)$.

The baseband waveform is $u(t) = ap(t)$ where $a = a_1 + ia_2$ with $a_1 = \Re\{a\}, a_2 = \Im\{a\}$.

The passband transmitted waveform is

$$x(t) = a_1\Psi_1(t) + a_2\Psi_2(t).$$

$$\Psi_1(t) = \Re\{2p(t)e^{2\pi i f_c t}\}; \qquad \Psi_2(t) = \Im\{2p(t)e^{2\pi i f_c t}\}$$

These two waveforms are orthogonal (in real vector space), each with energy 2. For $p(t)$ real, they are just $p(t)$ modulated by cosine and sine.

The received waveform is

$$Y(t) = (a_1+Z_1)\Psi_1(t) + (a_2+Z_2)\Psi_2(t) + Z'(t)$$

where $Z'(t)$ is real passband WGN in other degrees of freedom.

$$Y(t) = (a_1+Z_1)\Psi_1(t) + (a_2+Z_2)\Psi_2(t) + Z'(t)$$

**Since $\Psi_1$ and $\Psi_2$ are orthogonal and equal energy, $Z_1$ and $Z_2$ are iid Gaussian.**

**After translation of passband signal and noise to baseband,**

$$V(t) = [a_1+Z_1 + i(a_2+Z_2)]p(t) + Z''(t)$$

**$Z''(t)$ is the noise orthogonal (in complex vector space) to $p(t)$.**

**First consider detection in real vector space. Here $(a_1, a_2)$ represents the hypothesis and $Z_1, Z_2$ are iid Gaussian, $\mathcal{N}(0, N_0/2)$.**

$$V(t) = [a_1 + Z_1 + i(a_2 + Z_2)]p(t) + Z''(t)$$

**Let** $Y_1 = a_1 + Z_1$**,** $Y_2 = a_2 + Z_2$**.**

**Note that** $\int V(t)p^*(\tau - t)\,dt$ **is the output from a complex matched filter. Sampling this at** $\tau = 0$ **yields** $(Y_1 + iY_2)$**.**

**The components in an expansion** $Z_3, Z_4, \dots,$ **in an orthonormal expansion of** $Z''(t)$ **are also observable, although we will not need them.**

$Z_3, Z_4, \dots,$ are real Gaussian rv's and can also be viewed as passband rv's. Assume a finite number of these variables. For any two hypotheses, $a$ and $a'$, the likelihoods are

$$
\begin{aligned}
f_{\vec{Y}|H}(\vec{y} \mid a) &= \frac{1}{(\pi N_0)^k} \exp \sum_{j=1}^{2} \frac{-(y_j - a_j)^2}{N_0} + \sum_{j=3}^{2k} \frac{-z_j^2}{N_0} \\
f_{\vec{Y}|H}(\vec{y} \mid a') &= \frac{1}{(\pi N_0)^k} \exp \sum_{j=1}^{2} \frac{-(y_j - a'_j)^2}{N_0} + \sum_{j=3}^{2k} \frac{-z_j^2}{N_0} \\
\mathbf{LLR}(\vec{y}) &= \sum_{j=1}^{2} \frac{2y_j a_j - 2y_j a'_j}{N_0} = \sum_{j=1}^{2} \frac{2y_j(a_j - a'_j)}{N_0}
\end{aligned}
$$

As usual, ML decoding is minimum distance decoding.

**We can rewrite the likelihood as**

$$f_{\vec{Y}|H}(\vec{y} \mid a) = f(y_1 y_2 | a) f(z_3, z_4, \ldots, )$$

**So long as $Z_3, Z_4, \ldots$ are independent of $Y_1, Y_2$ and $H$, they cancel out in the LLR.**

**This is why it makes sense to use the WGN model - all we need in detection is the independence from the relevant rv's.**

**In other words, $(Y_1, Y_2)$ is a sufficient statistic and $Z_3, Z_4, \ldots,$ are irrelevant (so long as they are independent of $Y_1, Y_2, H$).**

**This is true for all pairwise comparisons between input signals.**

Now view this detection problem in terms of complex rv's. Let $\alpha = a - a'$.

$$\begin{aligned} \text{LLR}(\vec{y}) &= \sum_{j=1}^{2} \frac{2y_j(a_j - a'_j)}{N_0} = \sum_{j=1}^{2} \frac{2y_j\alpha_j}{N_0} \\ &= \frac{2\Re\{y\}\Re\{\alpha\} + \Im\{y\}\Im\{\alpha\}}{N_0} \\ &= \frac{2\Re\{y\alpha^*\}}{N_0} = \frac{2\Re\{\langle y, \alpha\rangle\}}{N_0} \end{aligned}$$

In real vector space, we project $\vec{y}$ onto $\vec{\alpha}$.

In complex space, 2-vectors become scalars, inner product needs real part to be taken.

The real part is a "further projection" of a complex number to a real number.

Now let's look at the general case in WGN. Must consider problem of real signals and noise for arbitrary modulation.

The signal set $\mathcal{A} = \{\vec{a}_1, \ldots, \vec{a}_M\}$, is a set of $k$-tuples.

$$\vec{a}_m = (a_{m,1}, \ldots, a_{m,k})^{\mathsf{T}}.$$

$\vec{a}_m$ is then modulated to

$$b_m(t) = \sum_{j=1}^{k} a_{m,j}\phi_j(t)$$

where $\{\phi_1(t), \ldots, \phi_k(t)\}$ is a set of $k$ orthonormal waveforms.

Successive signals are independent and mapped arbitrarily, using orthogonal spaces.

**Let** $\vec{X}(t) \in \{\vec{b}_1(t), \ldots, \vec{b}_M(t)\}$**. Then**

$$X(t) = \sum_{j=1}^{k} X_j \phi_j(t)$$

**where, under hypothesis** $m$**,**

$$X_j = a_{m,j} \quad \textbf{for } 1 \le j \le k$$

**Let** $\phi_{k+1}(t), \phi_{k+2}(t) \ldots$ **be an additional set of orthonormal functions such that the entire set** $\{\phi_j(t); j \ge 1\}$ **spans the space of real** $\mathcal{L}_2$ **wave-forms.**

$$Y(t) = \sum_{j=1}^{\ell} Y_j \phi_j(t) = \sum_{j=1}^{k} (X_j + Z_j)\phi_j(t) + \sum_{j=k+1}^{\ell} Z_j \phi_j(t).$$

$$Y(t) = \sum_{j=1}^{\ell} Y_j\phi_j(t) = \sum_{j=1}^{k} (X_j + Z_j)\phi_j(t) + \sum_{j=k+1}^{\ell} Z_j\phi_j(t).$$

**Assume** $\vec{Z} = \{Z_1, \dots, Z_k\}$ **are iid Gauss.** $\vec{Z}' = \{Z_{k+1}, \dots, \}$ **is independent of** $\vec{Z}$ **and of** $\vec{a}$**.**

$$f_{\vec{Y},\vec{Z}'|H}(\vec{y}, \vec{z}'|m) = f_{\vec{Z}}(\vec{y} - \vec{a}_m) f_{\vec{Z}'}(\vec{z}').$$

$$\Lambda_{m,m'} = \frac{f_{\vec{Z}}(\vec{y} - \vec{a}_m)}{f_{\vec{Z}}(\vec{y} - \vec{a}_{m'})}.$$

**The MAP detector depends only on** $\vec{Y}$**. The other signals and other noise variables are irrelevant.**

This says that detection problems reduce to finite dimensional vector problems; that is, signal space and observation space are for all practical purposes finite dimensional.

The assumption of independent noise and independent other signals is essential here.

With dependence, error probability is lowered; what you don't know can't hurt you.

## Detection for Orthogonal Signal Sets

We are looking at an alphabet of size $m$, mapping letter $j$ into $\sqrt{E}\phi_j(t)$ where $\{\phi_j(t)\}$ is an orthonormal set.

WGN of spectral density $N_0/2$ is added to the transmitted waveform.

The receiver gets

$$Y(t) = \sum_j Y_j\phi_j(t) = \sum_j (X_j + Z_j)\phi_j(t)$$

Only $\{Y_1, \dots, Y_m\}$ is relevant.

Under hypothesis $k$, $Y_k = \sqrt{E} + Z_j$ and $Y_j = Z_j$ for $j \neq k$.

**Use ML detection. Choose $k$ for which $\langle \vec{y}, \vec{x}_k \rangle$ is smallest.**

$$|\langle \vec{y}, \vec{x}_k \rangle|^2 = \sum_{j \neq k} y_j^2 + (y_k - n_k)^2 = \sum_{j=1}^{m} y_y^2 + E - 2\sqrt{E} y_k$$

**ML: choose $k$ for which $y_k$ is largest.**

**By symmetry, the probability of error is the same for all hypotheses so we look at hypothesis 1.**

**First scale outputs by $\sqrt{N_0/2}$, i.e., $W_j = Y_j \sqrt{2/N_0}$.**

**Under $H_1$, $W_1 \sim \mathcal{N}(\sqrt{2E/N_0}, 1)$ and $W_j \sim (0, 1)$ for $j \neq 1$.**

$$\Pr(e) = \int_{-\infty}^{\infty} f_{W_1|H}(w_1 \mid 1) \Pr\left( \bigcup_{j=2}^{m} (W_j \geq w_1 \mid 1) \right) dw_1$$

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