LECTURE 12

Last time:

- Strong coding theorem
- Revisiting channel and codes
- Bound on probability of error
- Error exponent

Lecture outline

- Error exponent behavior
- Expunging bad codewords
- Error exponent positivity
- Strong coding theorem

Last time

$$E_{codebooks}[P_{e,m}] \leq 2^{-N(E_0(\rho, P_X(x)) - \rho R)}$$

for

$$\begin{aligned} & E_0(\rho, P_X(x)) \\ = \; & -\log \left(\sum_y \left[\sum_{x_N} P_X(x) P_{Y|X}(y_i|x)^{\frac{1}{1+\rho}} \right]^{1+\rho} \right) \end{aligned}$$

We need to:

- get rid of the expectation over codes by throwing out the worst half of the codes
- Show that the bound behaves well (exponent is $-N\alpha$ for some $\alpha > 0$)
- Relate the bound to capacity

Error exponent

Define

$E_r(R) = \max_{0 \leq \rho \leq 1} \max_{P_X}(E_0(\rho, P_X(x)) - \rho R)$

then

$$E_{codebooks}[P_{e,m}] \leq 2^{-NE_r(R)}$$

$$E_{codebooks,messages}[P_e] \leq 2^{-NE_r(R)}$$

For a BSC:

Expunge codewords

The new $P_{e,m}$ is bounded as follows:

$$
\begin{aligned}
& P_{e,m} \\
= \;& 2.2^{-NE_r(\max_{0\leq\rho\leq 1}\max_{P_X}(E_0(\rho,P_X(x))-\rho\frac{\log(2M)}{N}))} \\
= \;& 2.2^{-NE_r(\max_{0\leq\rho\leq 1}\max_{P_X}(E_0(\rho,P_X(x))-\frac{\rho}{N}-\rho\frac{\log(M)}{N}))} \\
\leq \;& 4.2^{-NE_r(\max_{0\leq\rho\leq 1}\max_{P_X}(E_0(\rho,P_X(x))-\rho\frac{\log(M)}{N}))} \\
= \;& 4.2^{-NE_r(R)}
\end{aligned}
$$

Now we must consider positivity. Let $P_X(x)$ be such that $I(X;Y) > 0$, we'll show that the behavior of E_r is:

Error exponent positivity

We have that:

1. $E_0(\rho, P_X(x)) > 0, \forall \rho > 0$

2. $I(X;Y) \geq \frac{\partial E_0(\rho, P_X(x))}{\partial \rho} > 0, \forall \rho > 0$

3. $\frac{\partial^2 E_0(\rho, P_X(x))}{\partial \rho^2} \leq 0, \forall \rho > 0$

We can check that

$$I(X;Y) = \frac{\partial E_0(\rho, P_X(x))}{\partial \rho}|_{\rho=0}$$

then showing 3 will establish the LHS of 2

Showing the RHS of 2 will establish 1

Let us show 3

Error exponent positivity

To show concavity, we need to show that $\forall \rho_1, \rho_2 \; \forall \theta \in [0, 1]$

$$E_0(\rho_3, P_X(x))$$

$$\geq \theta E_0(\rho_1, P_X(x)) + (1 - \theta) E_0(\rho_2, P_X(x))$$

for $\rho_3 = \theta\rho_1 + \theta\rho_2$

We shall use the fact that

$$\frac{\theta(1+\rho_1)}{1+\rho_3} + \frac{(1-\theta)(1+\rho_2)}{1+\rho_3} = 1$$

and Hölder's inequality:

$$\sum_j c_j d_j \leq \left(\sum_j c_j^{\frac{1}{x}}\right)^x \left(\sum_j c_j^{\frac{1}{1-x}}\right)^{1-x}$$

Let us pick

$$
\begin{aligned}
c_j &= P_X(j)^{\frac{\theta(1+\rho_3)}{1+\rho_3}} P_{Y|X}(k|j)^{\frac{\theta}{1+\rho_3}} \\
d_j &= P_X(j)^{\frac{(1-\theta)(1+rho_2)}{1+\rho_3}} P_{Y|X}(k|j)^{\frac{1-\theta}{1+\rho_3}} \\
x &= \frac{\theta(1+\rho_1)}{1+\rho_3}
\end{aligned}
$$

Error exponent positivity

Proof continued

$$
\begin{aligned}
& \sum_{j \in \mathcal{X}} P_X(j) P_{Y|X}(k|j)^{\frac{1}{1+\rho_3}} \\
\leq \quad & \left(\sum_{j \in \mathcal{X}} P_X(j) P_{Y|X}(k|j)^{\frac{1}{1+\rho_1}} \right)^{\frac{\theta(1+\rho_1)}{1+\rho_3}} \\
\times \quad & \left(\sum_{j \in \mathcal{X}} P_X(j) P_{Y|X}(k|j)^{\frac{1}{1+\rho_2}} \right)^{\frac{(1-\theta)(1+\rho_2)}{1+\rho_3}} \\
\Rightarrow \quad & \left(\sum_{j \in \mathcal{X}} P_X(j) P_{Y|X}(k|j)^{\frac{1}{1+\rho_3}} \right)^{1+\rho_3} \\
\leq \quad & \left(\sum_{j \in \mathcal{X}} P_X(j) P_{Y|X}(k|j)^{\frac{1}{1+\rho_1}} \right)^{\theta(1+\rho_1)} \\
\times \quad & \left(\sum_{j \in \mathcal{X}} P_X(j) P_{Y|X}(k|j)^{\frac{1}{1+\rho_2}} \right)^{(1-\theta)(1+\rho_2)}
\end{aligned}
$$

Error exponent positivity

Proof continued

$$
\begin{aligned}
\Rightarrow \quad & \sum_{k\in\mathcal{Y}} \left(\sum_{j\in\mathcal{X}} P_X(j) P_{Y|X}(k|j)^{\frac{1}{1+\rho_3}} \right)^{1+\rho_3} \\
\leq \quad & \sum_{k\in\mathcal{Y}} \left(\sum_{j\in\mathcal{X}} P_X(j) P_{Y|X}(k|j)^{\frac{1}{1+\rho_1}} \right)^{\theta(1+\rho_1)} \\
\times \quad & \left(\sum_{j\in\mathcal{X}} P_X(j) P_{Y|X}(k|j)^{\frac{1}{1+\rho_2}} \right)^{(1-\theta)(1+\rho_2)} \\
\leq \quad & \left[\sum_{k\in\mathcal{Y}} \left(\sum_{j\in\mathcal{X}} P_X(j) P_{Y|X}(k|j)^{\frac{1}{1+\rho_1}} \right)^{(1+\rho_1)} \right]^{\theta} \\
\times \quad & \left[\left(\sum_{j\in\mathcal{X}} P_X(j) P_{Y|X}(k|j)^{\frac{1}{1+\rho_2}} \right)^{(1+\rho_2)} \right]^{(1-\theta)}
\end{aligned}
$$

Error exponent positivity

Proof continued

$$
\begin{aligned}
& -\log\left(\sum_{k\in\mathcal{Y}}\left(\sum_{j\in\mathcal{X}} P_X(j)P_{Y|X}(k|j)^{\frac{1}{1+\rho_3}}\right)^{1+\rho_3}\right) \\
\geq\ & -\theta\log\left(\sum_{k\in\mathcal{Y}}\left(\sum_{j\in\mathcal{X}} P_X(j)P_{Y|X}(k|j)^{\frac{1}{1+\rho_1}}\right)^{(1+\rho_1)}\right) \\
-\ & (1-\theta)\left(\left(\sum_{j\in\mathcal{X}} P_X(j)P_{Y|X}(k|j)^{\frac{1}{1+\rho_2}}\right)^{(1+\rho_2)}\right) \\
\Rightarrow\ & E_0(\rho_3, P_X) \geq \theta E_0(\rho_1, P_X) + (1-\theta)E_0(\rho_2, P_X)
\end{aligned}
$$

so E_0 is concave!

Error exponent positivity

Proof continued

Hence, extremum, if it exists, of $E_0(\rho, P_X) - \rho R$ over ρ occurs at $\frac{\partial E_0(\rho,P_X)}{\partial \rho} = R$, which implies that

$$\frac{\partial E_0(\rho,P_X)}{\partial \rho}\Big|_{\rho=1} \leq R \leq \frac{\partial E_0(\rho,P_X)}{\partial \rho}\Big|_{\rho=0} = I(X;Y)$$

is necessary for $E_r(R, P_X) = \max_{0 \leq \rho \leq 1}[E_0(\rho, P_X) - \rho R]$ to have a maximum

We have now placed mutual information somewhere in the expression

Critical rate is R_{CR} is defined as $\frac{\partial E_0(\rho,P_X)}{\partial \rho}\Big|_{\rho=1}$

Error exponent positivity

Proof continued

From $\frac{\partial E_0(\rho,P_X)}{\partial \rho} = R$

we obtain

$$\frac{\partial R}{\partial \rho} = \frac{\partial^2 E_0(\rho,P_X)}{\partial \rho^2}$$

Hence $\frac{\partial R}{\partial \rho} < 0$, R decreases monotonically from C to R_{CR}

We can write

$$E_r(R, P_X) = E_0(\rho, P_X) - \rho \frac{\partial E_0(\rho,P_X)}{\partial \rho}$$

for $E_r(R, P_X) = E_0(\rho, P_X) - \rho R$ (ρ allows parametric relation)

then

$$\frac{\partial E_r(R,P_X)}{\partial \rho} = -\rho \frac{\partial^2 E_0(\rho,P_X)}{\partial \rho^2} > 0$$

Error exponent positivity

Proof continued

Taking the ratio of the derivatives, $\frac{\partial E_r(R,P_X)}{\partial R} = -\rho$

$E_r(R, P_X)$ is positive for $R < C$

Moreover

$$\frac{\partial^2 E_r(R,P_X)}{\partial R^2} = -[\frac{\partial^2 E_0(\rho,P_X)}{\partial \rho^2}]^{-1} > 0$$

thus $E_r(R, P_X)$ is convex and decreasing in R over $R_{CR} < R < C$

Error exponent positivity

Proof continued

Taking $E_r(R) = \max_{P_X} E_r(R, P_X)$

is the maximum of functions that are convex and decreasing in R and so is also convex

For the P_X that yields capacity, $E_r(R, P_X)$ is positive for $R < C$

So we have positivity of error exponent for $0 < R < C$ and capacity has been introduced

This completes the coding theorem

Note: there are degenerate cases in which $\frac{\partial E_r(R,P_X)}{\partial \rho} = constant$ and $\frac{\partial^2 E_r(R,P_X)}{\partial^2 \rho} = 0$

when would that happen?