MASSACHUSETTS INSTITUTE OF TECHNOLOGY

6.436J/15.085J Fall 2018

Midterm

Boilerplate:

- No collaboration
- No internet, laptops, or cellphones
- Closed books, Closed notes.
- Cheat sheet allowed.
- Total: 100 pts

Exercise 1 (20 pts)**.** In honor of Laos' 65th independence day, some questions about independence.

1. Suppose A is independent of B, B is independent of C, and C is independent of A. Is $A \cap C$ necessarily independent of B?

   Prove or give counterexample

2. What if, in the above part, we also have that $B \cap C$ is independent of A?

   Prove or give counterexample

3. Suppose $A_1 \supset A_2 \supset A_3 \supset \dots$ and $A_n \to A$, and suppose that B is independent of every A_n. Is B necessarily independent of A?

   Prove or give counterexample

4. Suppose A is independent of *itself*. What does this say about $\mathbb{P}(A)$?

   Don't laugh - this actually happens and is crucial to prove certain very strong theorems, which we will go over later!

Bonus: No points, but a gold star for anyone who knows what country Laos gained independence from!

Exercise 2 15pts **.** Let $A_1, \dots, A_n$ be events. Let $X(\omega)$ be the number of events that occurred when ω was the elementary outcome. Show

$$\sum_{1 \le i < j \le n} \mathbb{P}[A_i \cap A_j] = \mathbb{E}\left[\frac{X(X-1)}{2}\right] .$$

Use this to prove

$$\mathbb{P}[\cup_{i=1}^n A_i] \geq \sum_{i=1}^n \mathbb{P}[A_i] - \sum_{1\leq i<j\leq n} \mathbb{P}[A_i \cap A_j]\,.$$

Hint: rewrite this as $\mathbb{E}[f(X)] \geq 0$.)

Exercise 3 15pts **.** Let X be a random variable taking values on non-negative integers $\mathbb{Z}_+ = \{0, 1, \ldots\}$. It has the following amazing property: There is a constant $c > 0$ such that for any bounded function $f : \mathbb{Z}_+ \to \mathbb{R}$ we have

$$\mathbb{E}[f(X)] = c\mathbb{E}[Xf(X-1)]\,.$$

Note: c does not depend on f.) Find distribution of X.

Exercise 4 20pts **.** Let X, Y be two independent standard normal random variables $\mathcal{N}(0,1)$. Let $Z = X^2 + Y^2$. Recall you don't need to prove it) that Z has pdf $f_Z(z) = \frac{1}{2}e^{-z/2}$, i.e. $Z \sim \text{Exp}(1/2)$.

1. Show that if U is independent of Z and uniform on $[0, 2\pi)$ then $\sqrt{Z}\sin mU$ is standard normal for any positive integer m.
2. Show that $T = \frac{2XY}{\sqrt{X^2+Y^2}}$ is standard normal. Hint: use polar coordinates

Exercise 5 30pts **.** Let $X_1, X_2, \ldots$ be a sequence of i.i.d. Bernoulli random variables coin tosses , such that $\mathbb{P}(X_1 = H) = p \in (0,1)$. Let

$$L_n = \max\{m \geq 0 : X_n = H,\, X_{n+1} = H,\, \ldots,\, X_{n+m-1} = H,\, X_{n+m} = T\}$$

be the length of the run of heads starting from the n-th coin toss. Prove that

$$\limsup_{n\to\infty} \frac{L_n}{\log(n)} = \frac{1}{\log(1/p)} \qquad \text{a.s.}. \qquad 1)$$

Steps:

1. Show that events $\{L_n \geq r\}, \{L_{n+r} \geq r\}, \{L_{n+2r} \geq r\}, \ldots$ are jointly independent.
2. Show that for any random variables Z_n

   $$\mathbb{P}[Z_n > \beta\text{-i.o.}] = 0 \quad \Rightarrow \quad \limsup Z_n \leq \beta \text{ a.s.}$$

   and

   $$\mathbb{P}[Z_n > \beta\text{-i.o.}] = 1 \quad \Rightarrow \quad \limsup Z_n \geq \beta \text{ a.s.}$$

3. Prove 1 . Hint: Borel-Cantelli