MASSACHUSETTS INSTITUTE OF TECHNOLOGY

6.436J/15.085J | Fall 2018

Problem Set 7

Readings:
Notes from Lectures 11-13.
[GS], Section 4.1-4.8 **and** 5.1-5.2
[Cinlar], Chapter IV.

Exercise 1. (Continuous-discrete Bayes rule) Let K be the number of heads obtained in six (conditionally) independent coins of a biased coin whose probability of heads is itself a random variable Z, uniformly distributed over $[0, 1]$. Find the conditional PDF of Z given K, and calculate $\mathbb{E}[Z \mid K = 2]$. You can use the following formula,

$$\int_0^1 y^{\alpha}(1-y)^{\beta}\,dy = \frac{\alpha\,!\beta\,!}{(\,\alpha+\,\beta+1)!},$$

known to be valid for positive integer α and β.

Exercise 2. Let X and Y be independent exponential random variables with parameter 1. Find the joint density function of $U = X+Y$ and $V = X/(X+Y)$, and deduce that V is uniformly distributed on $[0, 1]$.

Exercise 3. A point (X, Y) is picked at random uniformly in the unit circle. Find the joint density of R and X, where $R^2 = X^2 + Y^2$.

Exercise 4. Let X_1, X_2, X_3 be independent random variables, uniformly distributed on $[0, 1]$.

a. What is the probability that three rods of lengths X_1, X_2, X_3 can be used to make a triangle? (That is, that the largest one is smaller than the sum of the other two.)

b. What is the probability distribution of the second largest X_k, i.e. $X^{(2)}$.

Exercise 5. A stick is broken, at a location chosen uniformly at random. Find the average ratio of the lengths of the smaller and larger pieces.

Exercise 6. Let $X \sim \Gamma(a, c)$, $U, V \sim \Gamma(a, \sqrt{2c})$ and $Y \sim \mathcal{N}(0, 1)$, all jointly independent. Compare the distribution of $U - V$ and $\sqrt{X}Y$. (*Hint:* compute MGFs using conditional expectation).

Exercise 7. Let $X, Y \sim \Gamma(1, c)$ be independent and $Z = X + Y$. Describe conditional distribution $P_{Y|Z}$. (Ideally, you want to describe it as a Markov kernel $K(z, dy)$, however, full credit will be given for just specifying the conditional pdf or cdf).