Readings:
(a) Notes from Lectures 7-9.
(b) [Cinlar] Sections I.4-I.6

Exercise 1. The worker's union requests that all workers at a factory be given the day off if at least one worker has a birthday on that day. Otherwise workers agree to work 365 days a year. Management is to maximize the number of expected man-days worked per year. How many workers should they hire?

(Workers' birthdays are uniformly and independently distributed over the 365 days of the year.)

Exercise 2. Let $\Omega = \mathbb{Z}_+$, $\mathcal{F} = 2^\Omega$. Complete construction of a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and come up with a sequence of random variables X_n which is increasing a.e., but $\mathbb{E}[X_n]$ does not converge to $\mathbb{E}[X]$, where $X = \lim_n X_n$ a.e.

Exercise 3. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $X \geq 0$ a random variable. Show

$$\mathbb{E}[X] = \int_0^\infty (1 - F_X(x))dx \,,$$

where $F_X(x) = \mathbb{P}[X \leq x]$ is a CDF of X. (*Hint:* Fubini.)

Exercise 4. Show that for integrable f

$$\left| \int f d\mu \right| \leq \int |f| d\mu.$$

Exercise 5 (Weird integrable functions)**.** Let $\psi(x) = \frac{1}{\sqrt{x}} \mathbb{1}_{(0,1)}(x)$ and

$$F(x) = \sum_{n=1}^\infty 2^{-n} \psi(x - r_n) \,,$$

where $\{r_n\}$ is some enumeration of all rationals in $(0, 1)$. Show that $F(x)$ is a measurable non-negative function with

$$\int_{[0,1]} F d\lambda < \infty \,.$$

In particular, $F(x)$ is finite almost everywhere on $[0, 1]$, yet unbounded on every interval.

Exercise 6. For all n, let g_n and g be measurable functions. Suppose that $g_n \uparrow g$ and that $\int g_{1-} \; d\mu < \infty$ (where $g_{1-} = \max(-g_1, 0)$). Prove that $\int g_n \; d\mu \uparrow \int g \; d\mu$.

Exercise 7. (Differentiating under the integral sign)
Let $g : \mathbb{R}^2 \mapsto \mathbb{R}$ be a continuous function of two variables s and x. Furthermore, assume that the derivative $g'(s, x) = (\partial g / \partial s)$ exists for every s and x, is jointly measurable in (s, x) and is a continuous function of s for any fixed x. Assume $|g'(s, x)| \leq c$ for all s, x.

Let X be a random variable. Show that

$$\frac{\partial}{\partial s}\mathbb{E}[g(s, X)] = \mathbb{E}\left[\frac{\partial g}{\partial s}(s, X)\right].$$

Note: You can use the fact from elementary calculus that under our assumptions, $g(s, x) = g(0, x) + \int_0^s \frac{\partial g}{\partial s}(u, x) du$ for all x.