MASSACHUSETTS INSTITUTE OF TECHNOLOGY

6.436J/15.085J — Fall 2018

Problem Set 11

Exercise 1. A particle performs a random walk on the vertex set of a finite connected undirected graph G, which for simplicity we assume to have neither self-loops nor multiple edges. At each stage it moves to a neighbor of its current position, each such neighbor being chosen with equal probability. If G has η edges, show that the stationary distribution is given by $\pi_v = d_v/(2\eta)$, where d_v is the degree of each vertex v.

Exercise 2. A particle performs a random walk on a bow tie $ABCDE$ drawn on Figure 1, where C is the knot. From any vertex, its next step is equally likely to be to any neighbouring vertex. Initially it is at A. Find the expected value of:

(a) The time of first return to A.

(b) The number of visits to D before returning to A.

(c) The number of visits to C before returning to A.

(d) The time of first return to A, given that there were no visits to E before the return to A.

(e) The number of visits to D before returning to A, given that there were no visits to E before the return to A.

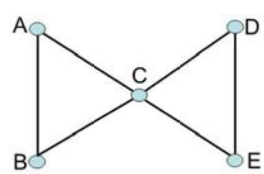

Figure 1: A bow tie graph.

Exercise 3. Let $(\Omega, \mathcal{F}) = (\mathbb{R}^\infty, \mathcal{B}^\infty)$, $X_k(\omega) = \omega_k$, $k \in \mathbb{N}$, be the canonical coordinate functions, and $\mathcal{F}_k = \sigma(X_1, .., X_k)$, $k \in \mathbb{N}$, be the natural filtration of this space. We say that τ is a stopping time of the filtration $\{\mathcal{F}_k\}$ if

(a) τ is a positive integer

(b) for every $k \geq 1$ we have $\{\tau \leq k\} \in \mathcal{F}_k$

Let $\tau : \Omega \to \mathbb{N}$ be $(\mathcal{F}, \mathcal{B})$ measurable. Show that τ is a stopping of $\{\mathcal{F}_k\}$ if and only if for every $\omega, \omega' \in \Omega$ and for every $n \geq 1$

$$\tau(\omega) = n,\ X_k(\omega) = X_k(\omega') \quad \forall\, 1 \leq k \leq n \quad \Rightarrow \quad \tau(\omega') = n. \tag{1}$$

Exercise 4. Let τ be a stopping time of a filtration $\mathcal{F}_n$. Recall that the σ-algebra $\mathcal{F}_\tau$ of "past until τ" is defined as

$$\mathcal{F}_\tau = \{E : E \cap \{\tau \leq n\} \in \mathcal{F}_n \quad \forall n\}$$

Show that for every random variable V measurable with respect to $\mathcal{F}_\tau$ there exists a stochastic process $\{G_n, n = 1, \ldots\}$, with G_n measurable with respect to $\mathcal{F}_n$, such that

$$V = G_\tau\,.$$

(Hint: First consider simple V).

Exercise 5. *(Cover time of C_n)* For a MC with state space $\mathcal{X}$ we define τ_{cov} to be the first time that every element of $\mathcal{X}$ was visited. The covering time $t_{cov} = \max_{x \in \mathcal{X}} \mathbb{E}^x[\tau_{cov}]$. Consider a MC that is a simple random walk on an n-cycle: it moves with probability $1/2$ to one of the neighbors each time. Show that $t_{cov}(n) = \frac{n(n-1)}{2}$ (Lovász'93). (Hint: Let τ_n be the first time a simple random walk on $\mathbb{Z}$ started at 0 visits n distinct states. Relate to t_{cov} and gambler's ruin.)

Exercise 6. *(Last visited vertex of C_n)* Consider a simple random walk X_t on an n-cycle C_n and let τ_{cov} be the first time that every vertex was visited. Show that given that $X_0 = v$ the distribution of $X_{\tau_{cov}}$ is uniform on $\{v\}^c$. (Hint: Notice that to have $X_{\tau_{cov}} = k$ the random walk should visit the states $k-1$ and $k+1$ before k.)
Fun fact: cycles and cliques are the only graphs with this property (Lovász-Winkler'93).

Exercise 7. Let B_k be iid with law $\mathbb{P}[B_k = +1] = p = 1 - \mathbb{P}[B_k = -1]$. Answer the following:

- Let $X_n = B_n B_{n+1}$, $n \geq 0$. Is it Markov? If yes, find its transition kernel.
- Let $Y_n = \frac{1}{2}(B_n - B_{n-1})$, $n \geq 1$. Is it Markov? If yes, find its transition kernel.

- Let $Z_n = |\sum_{k=1}^{n} B_k|$, $n \geq 1$. Is it Markov? If yes, find its transition kernel.
- If $\{V_i, i \geq 0\}$ is a Markov process with state space $\mathcal{X}$, and E_j are some subsets of $\mathcal{X}$, is it true that

  $$\mathbb{P}[V_n \in E_n | V_{n-1} \in E_{n-1}, V_{n-2} \in E_{n-2}, \dots, V_0 \in E_0] = \mathbb{P}[V_n \in E_n | V_{n-1} \in E_{n-1}],$$

  provided that $\mathbb{P}[V_{n-1} \in E_{n-1}, \dots, V_0 \in E_0] > 0$?
- Suppose that $P(x, y)$ is a kernel of an irreducible Markov chain. If $P(\cdot, x_1) = P(\cdot, x_2)$ show that $\pi(x_1) = \pi(x_2)$, where π is a stationary distribution. What if the chain is not irreducible?