MASSACHUSETTS INSTITUTE OF TECHNOLOGY

6.436J/15.085J — Fall 2018

Problem Set 10

Readings:
Notes from Lecture 18, 19 and 20.
[Cinlar] Chapter III.
[GS] Section 7.5, 7.10, 8.1-8.3.

Exercise 1. Let $S_n = \sum_{j=1}^{n} X_j$ be a sum of independent random variables X_j with $|X_j| \leq 1$ almost surely. Show that S_n converges in probability if and only if it converges almost surely (to a finite value).
(Hint: See how the case $\sum \mathrm{var}[X_j] = \infty$ was treated in the converse part of Kolmogorov-Khintchine in Lecture 19.)

Exercise 2. Let $\{X_n\}$ be a sequence of identically distributed random variables, with finite variance. Suppose that $\mathrm{cov}(X_i, X_j) \leq \alpha^{|i-j|}$, for every i and j, where $|\alpha| < 1$. Show that the sample mean $(X_1 + \cdots + X_n)/n$ converges to $\mathbb{E}[X_1]$, in probability.

Exercise 3. Given an i.i.d. sequence $X_n, n \geq 1$ with $\sigma^2 \triangleq \mathrm{var}(X_1) < \infty$, the CLT states that

$$\lim_{n\to\infty} \mathbb{P}\Big(\frac{\sum_{1\leq i\leq n} X_i - n\mathbb{E}[X_1]}{\sigma n^{\alpha}} \leq x\Big) = \Phi(x) \triangleq \int_{-\infty}^{x} \frac{e^{\frac{-t^2}{2}}}{\sqrt{2\pi}} dt,$$

when $\alpha = 1/2$. Compute the limit above for every $\alpha > 0$ and every x.

Exercise 4. Show that given an i.i.d. sequence $X_n, n \geq 1$ with mean μ, variance σ^2, while $(\sum_{1\leq i\leq n} X_i - \mu n)/(\sqrt{n}\sigma) \to N(0,1)$ in distribution, it is not the case that the same sequence converges in probability. (Hint: Cauchy criterion)

Exercise 5. Give an example of:

1. Independent zero-mean X_j's such that $\sum \mathrm{var} X_j$ diverges but

$$S_n = \sum_{k=1}^{n} X_j \qquad (1)$$

converges almost surely.

2. Independent zero-mean X_j taking values in $[-1, 1]$ such that $X_j \stackrel{\text{a.s.}}{\to} 0$ but S_n does not converge almost surely.

Exercise 6. Let $\{X_n\}$ be a sequence of nonnegative integrable random variables and X an integrable random variable. Suppose $X_n \stackrel{\text{a.s.}}{\to} X$ and $\mathbb{E}[X_n] \to \mathbb{E}[X]$. Show that the family $\{X_n, n = 1, \ldots\}$ is uniformly integrable. Conclude that $X_n \stackrel{L_1}{\to} X$, i.e.

$$\mathbb{E}[|X_n - X|] \to 0$$

(Thus, $Y_n \stackrel{\text{a.s.}}{\to} Y$ is u.i. iff $\mathbb{E}[|Y_n|] \to \mathbb{E}[|Y|]$.)

Exercise 7. Let $N(\cdot)$ be a Poisson process with rate λ. Find the covariance of $N(s)$ and $N(t)$.

Exercise 8. Based on your understanding of the Poisson process, determine the numerical values of a and b in the following expression and explain your reasoning.

$$\int_t^\infty \frac{\lambda^5 \tau^4 e^{-\lambda\tau}}{4!}\, d\tau = \sum_{k=a}^{b} \frac{(\lambda t)^k e^{-\lambda t}}{k!}.$$

Exercise 9. ***(practice problem, not for grade)***

(a) Shuttles depart from New York to Boston every hour on the hour. Passengers arrive according to a Poisson process of rate λ per hour. Find the expected number of passengers on a shuttle. (Ignore issues of limited seating.)

(b) Now, and for the rest of this problem, suppose that the shuttles are not operating on a deterministic schedule, but rather their interdeparture times are exponentially distributed with rate μ per hour, and independent of the process of passenger arrivals. Find the PMF of the number shuttle departures in one hour.

(c) Let us define an "event" in the terminal to be either the arrival of a passenger, or the departure of a shuttle. Find the expected number of "events" that occur in one hour.

(d) If a passenger arrives at the gate, and sees 2λ people waiting, find his/her expected time to wait until the next shuttle.

(e) Find the PMF of the number of people on a shuttle.