MASSACHUSETTS INSTITUTE OF TECHNOLOGY

6.436J/15.085J — Fall 2008

Problem Set 5 — due 10/10/2008

Readings: Notes for lectures 8 and 9

Recommended readings: Chapter 3 of [BT]; and Sections 4.1-4.6 of [GS].

Exercise 1. Let X_1, X_2, X_3 be independent random variables, uniformly distributed on [0,1]. Let Y be the median of X_1, X_2, X_3 (that is the middle of the three values). Find the conditional CDF of X_1, given the event $Y = 1/2$. Under this conditional distribution, is X_1 continuous? Discrete?

Exercise 2. One of two wheels of fortune, A and B, is selected by the toss of a fair coin, and the wheel chosen is spun once to determine the value of a random variable X. If wheel A is selected, the PDF of X is

$$f_{X|A}(x|A) = \begin{cases} 1 & \text{if } 0 < x \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

If wheel B is selected, the PDF of X is

$$f_{X|B}(x|B) = \begin{cases} 3 & \text{if } 0 < w \leq 1/3, \\ 0 & \text{otherwise.} \end{cases}$$

If we are told that the value of X was less than $1/4$, what is the conditional probability that wheel A was the one selected?

Exercise 3. Let X and Y be positive random variables. Construct an example to show that it is possible to have $\mathbb{E}[X \mid Y] > Y$ and $\mathbb{E}[Y \mid X] > X$, with probability 1, and explain why this does not contradict the facts $\mathbb{E}[\mathbb{E}[X \mid Y]] = \mathbb{E}[X]$ and $\mathbb{E}[\mathbb{E}[Y \mid X]] = \mathbb{E}[Y]$.

Hint: Let N be a geometric random variable with parameter p, and fix some $m > 0$. Let $(X, Y) = (m^N, m^{N-1})$ or $(X, Y) = (m^{N-1}, m^N)$, with equal probability.

Exercise 4. Let $X \sim N(0, 1)$, and $\Phi(x) = \mathbb{P}(X \leq x)$.

(a) Show that

$$(x^{-1} - x^{-3})e^{-x^2/2} < \sqrt{2\pi}[1 - \Phi(x)] < x^{-1}e^{-x^2/2}, \qquad x > 0.$$

(b) Let X be $N(0,1)$, and $a > 0$. Show that $\mathbb{P}(X > x + a/x \mid X > x) \to e^{-a}$, as $x \to \infty$.

Exercise 5. Let X_1, X_2, X_3 be independent variables, uniformly distributed on $[0,1]$. What is the probability that three rods of lengths X_1, X_2, X_3 can be used to make a triangle?

Exercise 6. We have a stick of unit length $[0,1]$, and break it at X, where X is uniformly distributed on $[0,1]$. Given the value x of X, we let Y be uniformly distributed on $[0,x]$, and let Z be uniformly distributed on $[0,1-x]$. We assume that conditioned on $X = x$, the random variables Y and Z are independent. Find the joint PDF of Y and Z. Find $\mathbb{E}[X|Y]$, $\mathbb{E}[X|Z]$, and $\rho(Y,Z)$.

Exercise 7. Let α and β be positive integers. The following formula is known to be true and can be established using integration by parts:

$$\int_0^1 y^\alpha (1-y)^\beta \, dy = \frac{\alpha!\, \beta!}{(\alpha+\beta+1)!}. \tag{1}$$

The objective of this problem is to provide a probabilistic derivation of this formula (that is, you are not allowed to use integration by parts or other integration tricks), and then to use this formula in an inference problem.

(a) Prove (1) using a probabilistic argument.

Hint: Let $Y, Y_1, \ldots, Y_{\alpha+\beta}$ be independent random variables, uniformly distributed on $[0,1]$. Start by writing down a formula for the probability of the event

$$\big\{ \max\{Y_1,\ldots,Y_\alpha\} \le Y \text{ and } Y \le \min\{Y_{\alpha+1},\ldots,Y_{\alpha+\beta}\}\big\}.$$

(b) Let K be the number of heads obtained in six independent coins of a biased coin whose probability of heads is itself a random variable Z, uniformly distributed over $[0,1]$. Find the conditional PDF of Z given K, and calculate $\mathbb{E}[Z \mid K = 2]$.

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