6.302 Feedback Systems

Recitation 23: Time Delays

Prof. Joel L. Dawson

Today we're going to talk about time delays, affectionately referred to as "e^{-st}" in the Laplace domain. Why e^{-st}? Recall:

$$h(t) \rightarrow \boxed{\text{time delay}} \rightarrow h(t-T)$$

Laplace transform:

$$\mathcal{L}\{h(t-T)\} = \int_{-\infty}^{\infty} h(t-T)e^{-st}d\tau$$

Let $t' = t - T$

$$dt' = dt \rightarrow \mathcal{L}\{\cdot\} = \int_{-\infty}^{\infty} h(t')\, e^{-s(t'+T)}\, dt'$$
$$= e^{-sT} \int_{-\infty}^{\infty} h(t')\, e^{-st'}\, dt'$$
$$= e^{-sT}\, H(s)$$

Pure delays are tough to deal with in control systems. (Remember the car with the delay in the steering column?) Let's see if we can understand why with our newfound knowledge of feedback theory.

$\boxed{\text{CLASS EXERCISE}}$

Suppose you had an uncompensated system with loop transmission

$$L(s) = \frac{10^4}{s} e^{-s\left(\frac{1}{10^4} \cdot \left(\frac{\pi}{2} \cdot 100\right)\right)}$$

1) What is the crossover frequency ω_c?

2) What is the phase at crossover?

3) What would <u>you</u> do to arrange for 90° of phase margin @ ω_c?

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Delays are tough. They unload tons of negative phase without doing us the courtesy of dropping the magnitude. If we add a pole to lower the magnitude, we pick up -90˚ of phase. If we add a zero to increase the phase, the magnitude goes up.

How do we deal with this in practice? We don't. Very often, we opt for dominant pole compensation and just make sure ω_c occurs before the delay contributes significant phase shift.

In our example, where would ω_c have to occur in order to get 45˚ of pm? We've already got our dominant pole, an integrator, which gives us -90˚ of phase. We must figure out where the delay gives only -45˚ ($-\frac{\pi}{4}$) of phase:

$$-\omega_c \left(\frac{1}{10^4} \cdot \frac{\pi}{2} \cdot 100\right) = -\frac{\pi}{4}$$

$$\omega_c \cdot \frac{\cancel{\pi}}{2} = \frac{\cancel{\pi}}{4} \cdot 100$$

$$\omega_c = 50 \text{ rps}$$

So make L(s) equal to:

$$L(s) = \frac{50}{s} e^{-s\left(\frac{1}{10^4} \cdot \left(\frac{\pi}{2} \cdot 100\right)\right)}$$

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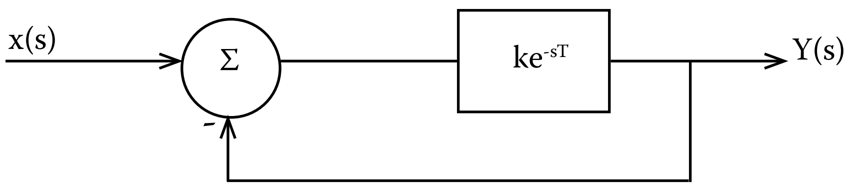
How else can we look at delays? It turns out that we can look at them as a very, very large collection of poles. Suppose we have the following system:



$$\frac{Y(s)}{x(s)} = \frac{ke^{-sT}}{1 + ke^{-sT}}$$

To figure out where the poles are, we rely on the characteristic equation $1 + ke^{-sT} = 0$

$$ke^{-sT} = -1$$

$$ke^{-sT} = e^{\,j(2n+1)\pi} \qquad (n = \{\cdots -3, -2, -1, 0, 1, 2, 3, \cdots\})$$

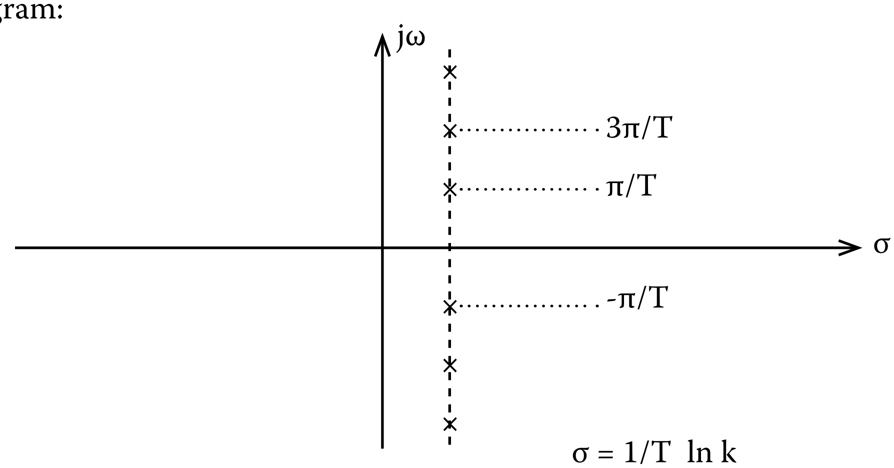
$$\ln [\; ke^{-sT} = e^{\,j(2n+1)\pi} \;]$$

$$\ln(\, k - sT) = j(2n+1)\pi$$

$$-sT = -\ln k + j(2n+1)\pi$$

$$s = \frac{1}{T}\ln k - j(2n+1)\,\frac{\pi}{T}$$

Pole-zero diagram:



Note we can make this system stable by choosing k such that these poles are in the LHP:

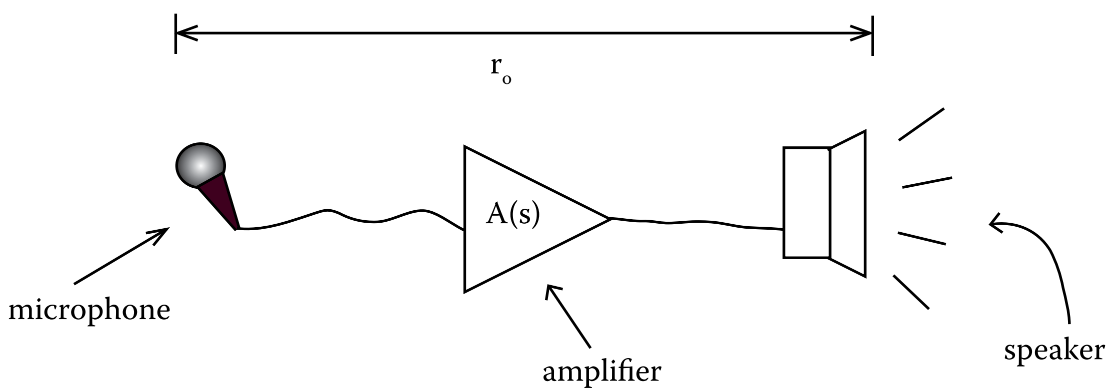
$$\frac{1}{T}\ \ln k < 0$$

$$\ln k < 0$$

$$k < 1$$

$\Rightarrow$ system is "crossed over" for all frequencies.

Being familiar with delay helps us to model a physical situation that we have all witnessed: acoustic feedback.



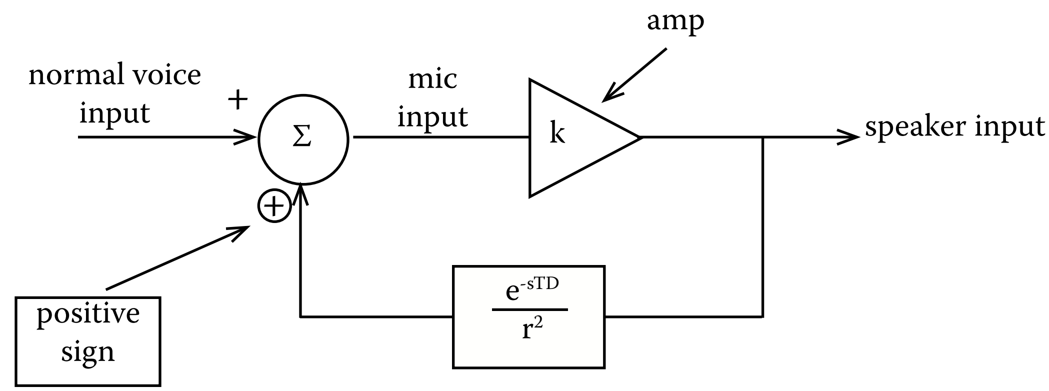
A(s): Typically a low-pass function of some sort. For now, though, let's say that it provides a frequency-independent gain k.

Delay: Sound travels at a speed of v_o meters/sec, and the microphone is r_o meters away from the speaker. The time delay is thus:

$$T_D = \frac{r_o}{v_o}$$

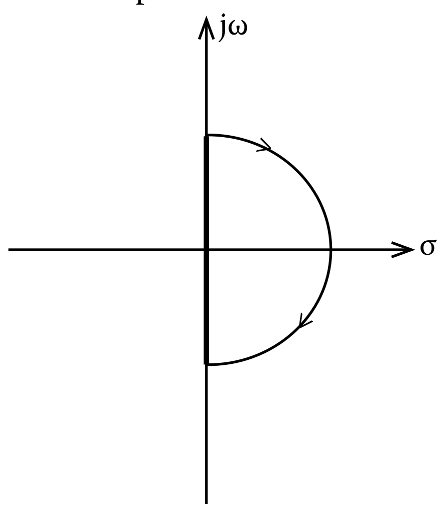
Moreover, sound amplitude decays in an inverse square manner with distance $\rightarrow \dfrac{1}{r^2}$

This is all we need to do our modeling. $\Rightarrow$

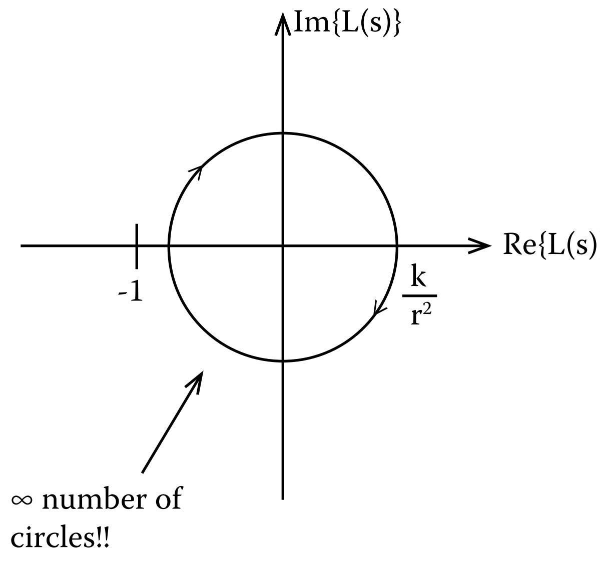


We can use Nyquist to figure out when things will go nuts.

$$L(s) = \frac{ke^{-sTD}}{r^2}$$



"D contour"



$Z = N + P \rightarrow Z = N$

$p = 0$

So, we get an infinite number of encirclements of the -1 point when

$$\frac{k}{r^2} > 1$$

This tells us what we already know by experience. If we don't want the howling, we should:

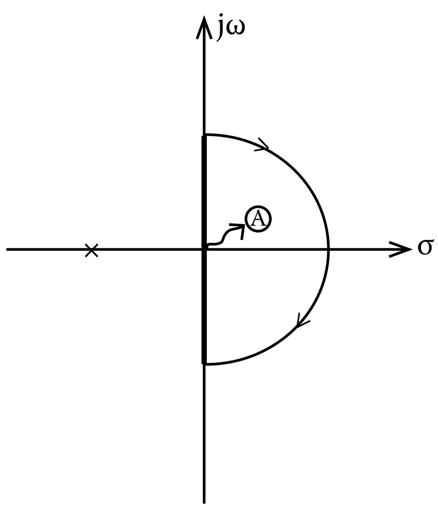
1) Lower k → turn down the volume

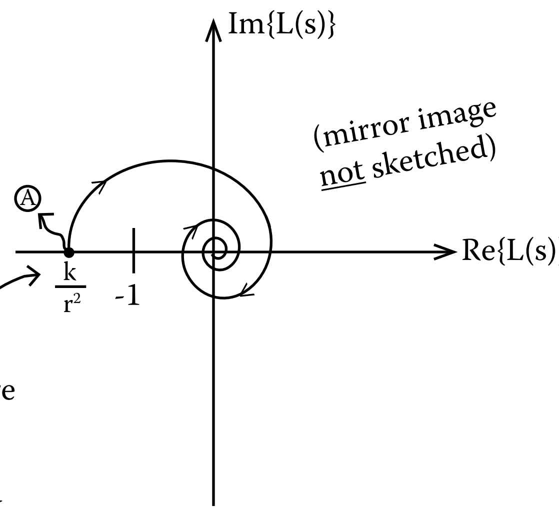
2) Raise r_o → move the microphone away from the speaker

It turns out that dynamics in the amplifier don't change the qualitative results. Suppose A(s) was a single pole amplifier:

$$A(s) = \frac{k}{\tau s + 1}$$

So that L(s) is now $\frac{k}{r^2} \quad \frac{e^{-sTD}}{\tau s + 1}$. Nyquist gives





Still, to avoid encirclements we either lower the volume or move the microphone away from the speaker.