MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Electrical Engineering and Computer Science

6.262   Discrete Stochastic Processes
Midterm Exam – Revised Wording
April 7, 2009

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Explanations and clarifications given on the board during the exam are included in this version in bold.
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There are 3 questions, each with several parts. If any part of any question is unclear to you, please ask.

The blue books are for scratch paper only. Don't hand them in. Put your final answers in the white booklets and briefly explain your reasoning for every question. **Please put your name on each white booklet you turn.**

Few questions require extensive calculations and most require very little, provided you pick the right tool or model in the beginning. The best approach to each problem is to first think carefully over what you've learned and decide precisely what tool fits best - before putting pencil to paper.

Partial Credit

We will give partial credit if you present your thinking in a clear way we can understand (and your thinking is at least partially correct), but otherwise not. If you model a problem using a tool that requires significant computation, it is best to first give the model explicitly and indicate how you will use the results of the computation to determine the final answer. This approach will help you receive fair credit if your computations aren't perfect.

I. (24 pts) Shorter Questions

a) For the following finite-state Markov chains, each transition is marked ⇔ the transition probability is nonzero. For each chain, identify all classes, determine the period of each class, and specify whether each class is recurrent or transient.

(i)
4 pts

(ii)
4 pts

(iii)
4 pts

b) **(6 pts)** A fair coin is tossed repeatedly. Find the expected number of tosses until the pattern HHTHTHTT is first seen. Please explain your reasoning.

c) **(6 pts)** Let $N_1(t)$ be a Poisson process with interarrival intervals X_k and a very rapid rate f. Let $N_2(t)$ be an independent Poisson process with interarrival intervals Y_k and a very slow rate $s \ll f$. Let the merged process $N_3(t) = N_1(t) + N_2(t)$ have interarrival intervals Z_k.

Given that the $(n-1)$st arrival of N_3 occurs at time t and the next arrival of N_3 comes from N_2, find the probability distribution for the next interarrival interval Z_n of N_3. Please explain your reasoning.

II. (36 pts) Bob and Alice work at the front desk of a store. Cheerful Alice deals with customers while Bob stocks the shelves and barks at suppliers over the phone. While Alice deals with any customer, the other customers stand in line and await their turn.

Every time a customer begins waiting to talk with Alice because she is dealing with a previous customer, Bob immediately jumps in to help. Without delay, Bob takes the first customer in the line, and he continues serving customers until no one is left waiting to speak with Alice. **However (to Alice's annoyance) every time Alice finishes with a customer while Bob is still helping someone, if no one else is waiting Bob immediately turns his customer over to Alice.** (Therefore 1 customer is waiting in line $\Leftrightarrow$ 3 customers are seeking or receiving service. **And there is 1 customer in service and no one in line $\Leftrightarrow$ Alice is serving a customer and Bob is not.**)

Customers arrive all day as a Poisson process with an average arrival rate of $\lambda = 3$ customers every 10 minutes. Bob and Alice each have an exponential service time, with an average service rate of $\mu = 2$ customers every 10 minutes. Service times are all iid and independent of arrivals.

a) **(6 pts)** Draw and label a sampled-time Markov chain description of the system. Use a sampling interval $\delta \ll 10$ minutes. Assume δ is sufficiently small that the likelihood of more than one arrival or departure (or an arrival and a departure) during δ is negligible. **Bring your carefully labelled diagram up for us to correct in real time, since any errors here can propagate to the remaining parts of the problem.**

b) **(6 pts)** Is the chain transient, null recurrent, or positive recurrent? Is it periodic or aperiodic? Please explain your reasoning.

c) **(6 pts)** The store starts out empty each morning. How long (in minutes) can Bob expect to drink coffee until he is first required to help Alice serve customers? Please explain your reasoning.

d) **(6 pts)** After the first period that Bob spends helping Alice is over, how long (in minutes) can he expect to be free to deal with other issues before she first needs his help again? Please explain your reasoning.

e) **(6 pts)** Over the entire work day, what fraction of the day will Bob spend helping Alice? Please explain your reasoning.

f) **(6 pts)** Find the expected length of any period **(in minutes)** Bob spends helping Alice. Please explain your reasoning.

III. **(40 pts)** A small production facility builds widgets. Widgets require two subassemblies, aidjets and bidgets. The time to build an aidjet is a nonnegative rv with density $f_a(t)$ and distribution function $F_a(t)$. Successive aidgets require IID construction intervals. The time to build a bidget is also a nonnegative rv with density $f_b(t)$ and distribution function $F_b(t)$. Successive bidget times are also IID. Also aidjet and bidget times are independent of each other. In **this question**, you can either choose f_a and f_b to be uniform over (0,2] and calculate a numerical answer, or else leave them abstract and provide a formula.

a) Initially the facility is set up with an aidget factory and a bidget factory but no storage. Thus construction of the first aidget and the first bidget starts at time 0, but whichever factory finishes first stops and waits until the other is finished. The widget is then produced in zero extra time and each factory starts on the next part. This continues ad infinitum. Let $N_1(t)$ be the number of widgets produced by time t.

      a1) **(5 pts)** Is $N_1(t)$ a renewal counting process?

      a2) **(5 pts)** Find the time-average number of widgets produced in the limit $t \to \infty$ and state carefully what that time-average means.

b) Now some storage is provided and two aidgets are produced one after the other and, starting at the same time, two bidgets are produced one after the other. Whichever factory finishes a pair first stops and waits for the other to finish a pair. The first widget is produced when both have finished one part and the second widget when both have finished the second part. When both finish their second part, both start again, and this continues ad infinitum. Let $N_2(t)$ be the number of widgets assembled by time t in this new scheme.

      b1) **(6 pts)** Is $N_2(t)$ a renewal counting process? If not, describe a renewal counting process for widgets in this scheme.

      b2) **(6 pts)** Show that $N_1(t) \le N_2(t)$. $\big($Hint: this is not an asymptotic result – look at the first pair of widgets. **Let A_n be the time it takes to produce the n-th aidget and B_n be the time it takes to produce n-th bidget. Assume these are the same random variables in parts a) and b), i.e., assume that the random vector $\omega \to A_1(\omega),\ A_2(\omega),\ \cdots$ is the same function of ω for parts a) and b), and the random vector $\omega \to B_1(\omega),\ B_2(\omega),\ \cdots$ is the same function of ω for parts a) and b) of this problem.**$\big)$

b3) **(6 pts)** Find the time-average number of widgets produced in the limit $t \to \infty$ and state carefully what that time-average means.

c) Now assume that neither facility ever stops and waits; they continue producing aidgets and bidgets, which are paired as available and immediately are combined into widgets. Let $N_3(t)$ be the number of widgets produced by time t.

c1) **(6 pts)** Explain carefully why $N_3(t)$ is not a renewal counting process.

c2) **(6 pts)** Find the time-average number of widgets produced in the limit $t \to \infty$ and state carefully what that time-average means.

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