Solutions to Homework 6

6.262 Discrete Stochastic Processes

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Exercise 1

Let $\{Y_n; n \geq 1\}$ be a sequence of rv's and assume that $\lim_{n\to\infty} \mathbb{E}[|Y_n|] = 0$. Show that $\{Y_n; n \geq 1\}$ converges to 0 in probability. Hint 1: Look for the easy way. Hint 2: The easy way uses the Markov inequality.

Solution: Applying the Markov inequality to $|Y_n|$ for arbitrary n and arbitrary $\epsilon > 0$, we have

$$\Pr\{|Y_n| \geq \epsilon\} \leq \frac{\mathbb{E}[|Y_n|]}{\epsilon}$$

Thus going to the limit $n \to \infty$ for the given ϵ,

$$\lim_{n\to\infty} \Pr\{|Y_n| \geq \epsilon\} = 0.$$

Since this is true for every $\epsilon > 0$, this satisfies the definition for convergence to 0 in probability.

Exercise 2 (4.2 in text)

The purpose of this exercise is to show that, for an arbitrary renewal process, $N(t)$, the number of renewals in $(0, t]$, has finite expectation.

a) Let the inter-renewal intervals have the distribution $\mathsf{F}_X(x)$, with, as usual, $\mathsf{F}_X(0) = 0$. Using whatever combination of mathematics and common sense is comfortable for you, show that numbers $\epsilon > 0$ and $\delta > 0$ must exist such that $\mathsf{F}_X(\delta) \leq 1 - \epsilon$. In other words, you are to show that a positive rv must take on some range of of positive values with positive probability.

Solution: For any $\epsilon < 1$, we can look at the sequence of events $\{X \geq 1/k; k \geq 1\}$. The union of these events is the event $\{X > 0\}$. Since $\Pr\{X \leq 0\} = 0$, $\Pr\{X > 0\} = 1$. The events $\{X \geq 1/k\}$ are nested in k, so that, from (**??**),

$$1 = \Pr \bigcup_k \{X \geq 1/k\} = \lim_{k\to\infty} \Pr\{X \geq 1/k\}$$

Thus, for k large enough, $\Pr\{X \geq 1/k\} \geq 1 - \epsilon$. Taking δ to be $1/k$ for that value of k completes the demonstration.

b) Show that $\Pr\{S_n \leq \delta\} \leq (1 - \epsilon)^n$.

Solution: S_n is the sum of n interarrival times, and, bounding very loosely, $S_n \leq \delta$ implies that for each $i, 1 \leq i \leq n$, $X_i \leq \delta$. The X_i are independent, so, since $\Pr X_i \leq \delta \leq (1 - \epsilon)$, we have $\Pr\{S_n \leq \delta\} \leq (1 - \epsilon)^n$.

c) Show that $\mathbb{E}[N(\delta)] \leq 1/\epsilon$.

Solution: Since $N(t)$ is nonnegative,

$$\begin{aligned}
\mathbb{E}[N(\delta)] &= \sum_{n=1}^{\infty} \Pr\{N(\delta) \geq n\} \\
&= \sum_{n=1}^{\infty} \Pr\{S_n \leq \delta\} \\
&\leq \sum_{n=1}^{\infty} (1-\epsilon)^n \\
&= \frac{1-\epsilon}{1-(1-\epsilon)} \;=\; \frac{1-\epsilon}{\epsilon} \;\leq\; \frac{1}{\epsilon}
\end{aligned}$$

d) Show that for every integer k, $\mathbb{E}[N(k\delta)] \leq k/\epsilon$ and thus that $\mathbb{E}[N(t)] \leq \frac{t+\delta}{\epsilon\delta}$ for any $t > 0$. **Solution:** The solution of part c) suggests breaking the interval $(0, k\delta]$ into k intervals each of size δ. Letting $N_i = N((i) - N(i-1)$ be the ith of these intervals, we have $\mathbb{E}[N(\delta k)] = \sum_{i=1}^{k} \mathbb{E}[N_i]$.

For the first of these intervals, we have shown that $\mathbb{E}[N_1] \leq 1/\epsilon$, but that argument does not quite work for the subsequent intervals, since the first arrival in that interval might be at the end of an interarrival interval greater than δ. All the other arrivals in that interval must still be at the end of an interarrival interval at most δ. Thus if let $S_n^{(i)}$ be the number of arrivals in the ith interval, we have

$$\Pr\{S_n^{(i)\}\leq\delta} \leq (1-\epsilon)^{n-1}$$

Repeating the argument in part c), then,

$$\begin{aligned}
\mathbb{E}[N_i] &= \sum_{n=1}^{\infty} \Pr\{N_i \geq n\} \;=\; \sum_{n=1}^{\infty} \Pr\{S_n^{(i)} \leq \delta\} \\
&\leq \sum_{n=1}^{\infty} (1-\epsilon)^{n-1} \;=\; \frac{1}{1-(1-\epsilon)} \;=\; \frac{1}{\epsilon}
\end{aligned}$$

Since $\mathbb{E}[N(\delta k)] = \sum_{i=1}^{k} \mathbb{E}[N_i]$, we then have

$$\mathbb{E}[N(k\delta)] \leq k/\epsilon$$

Since $N(t)$ is non-decreasing in t, it can be upper bounded by the integer multiple of $1/\delta$ that is just larger than t, i.e.,

$$\mathbb{E}[N(t)] \;\leq\; \mathbb{E}[N(\delta\lceil t/\delta\rceil)] \;\leq\; \frac{\lceil t/\delta\rceil)}{\epsilon} \;\leq\; \frac{t/\delta + 1}{\epsilon}$$

e) Use your result here to show that $N(t)$ is non-defective.

Solution: Since $N(t)$, for each t, is non-negative and has finite expectation, it certainly must be non-defective. One way to see this is that $\mathbb{E}[N(t)]$ is the integral of the complementary distribution function, $\mathsf{F}^{\mathsf{c}}_{N(t)}(n)$ of $N(t)$. Since this integral is finite, $\mathsf{F}^{\mathsf{c}}_{N(t)}(n)$ must approach 0 with increasing n.

3) Exercise 4.4 in text

Is it true for a renewal process that:
a) $N(t) < n$ if and only if $S_n > t$?
b) $N(t) \leq n$ if and only if $S_n \geq t$?
c) $N(t) > n$ if and only if $S_n < t$?
Solution: Part a) is true, as pointed out in (4.1). It is simply the complement of the statement that $N(t) \geq n$ if and only if $S_n \leq t$.

Parts b) and c) are false, as seen by any situation where $S_n < t$ and $S_{n+1} > t$. In these cases, $N(t) = n$.

4) Exercise 4.5 in text

(This shows that convergence WP1 implies convergence in probability.) Let $\{Y_n;\, n \geq 1\}$ be a sequence of rv's that converges to 0 WP1. For any positive integers m and k, let

$$A(m,k) = \{\omega : |Y_n(\omega)| \leq 1/k \quad \text{for all } n \geq m\}.$$

a) Show that if $\lim_{n\to\infty} Y_n(\omega) = 0$ for some given ω, then (for any given k) $\omega \in A(m,k)$ for some positive integer m.

Solution: Note that for a given ω, $\{Y_n(\omega);\, n \geq 1\}$ is simply a sequence of real numbers. Thus by the definition of convergence of a sequence of real numbers, if $\lim_{n\to\infty} Y_n(\omega) = 0$ then for every integer $k \geq 1$, there is an m such that $|Y_n(\omega)| \leq 1/k$ for all $n \geq m$. Thus the given ω must be in the set $A(m,k)$ for that k.

b) Show that for all $k \geq 1$

$$\Pr \bigcup\nolimits_{m=1}^{\infty} A(m,k) = 1.$$

Solution: We saw in a) that, given $k \geq 1$ and given $\lim_{n\to\infty} Y_n(\omega) = 0$, there is some $m \geq 1$ such that $\omega \in A(m,k)$. Thus given $\lim_{n\to\infty} Y_n(\omega) = 0$, $\omega \in \bigcup_m A(m,k)$

c) Show that, for all $m \geq 1$, $A(m,k) \subseteq A(m{+}1,\, k)$. Use this (plus (1.9) to show that

$$\lim_{m\to\infty} \Pr A(m,k) = 1.$$

Solution: If $\omega \in A(m,k)$, then $|Y_n(\omega)| \leq 1/k$ for all $n \geq m$ and thus for all $n \geq m+1$. Thus $\omega \in A(m,k)$ implies that $\omega \in A(m+1,k)$. This means that $A(m,k) \subseteq A(m+1,k)$. From (1.9) then

$$1 = \Pr \bigcup_m A(m,k) = \lim_{m\to\infty} \Pr A(m,k)$$

d) Show that if $\omega \in A(m,k)$, then $|Y_m(\omega)| \leq 1/k$. Use this (plus part c) to show that

$$\lim_{m\to\infty} \Pr |Y_m| > 1/k = 0.$$

Solution: By the definition of $A(m,k)$, $\omega \in A(m,k)$ means that $|Y_n(\omega)| \leq 1/k$ for all $n \geq m$, and thus certainly $|Y_m(\omega)| \leq 1/k$. Since $\lim_{m\to\infty} \Pr A(m,k) = 1$, it follows that

$$\lim_{m\to\infty} |Y_m(\omega)| \leq 1/k$$

Since $k \geq 1$ is arbitrary, this shows that $\{Y_n;\, n \geq 1\}$ converges in probability.

5) Exercise 4.8 in text:

a) Since $\mathbb{E}[X] = \int_0^\infty \mathrm{d}F_X^c(x)\, dx = \infty$, we know from the definition of an integral over an infinite limit that

$$\mathbb{E}[X] = \lim_{b\to\infty} \int_0^b \mathrm{d}F_X^c(x)\, dx = \infty$$

For $\breve{X} = \min(X, b)$, we see that $\mathrm{d}F_{\breve{X}}(x) = \mathrm{d}F_X(x)$ for $x \leq b$ and $\mathrm{d}F_{\breve{X}}(x) = 1$ for $x > b$. Thus $\mathbb{E}[\breve{X}] = \int_0^b \mathrm{d}F_X^c(x)\, dx$. Since $\mathbb{E}[\breve{X}]$ is increasing with b toward ∞, we see that for any $M > 0$, there is a b sufficiently large that $\mathbb{E}[\breve{X}] \geq M$.

b) Note that $X - \breve{X}$ is a non-negative rv, i.e., it is 0 for $X \leq b$ and greater than b otherwise. Thus $\breve{X} \leq X$. It follows then that for all $n \geq 1$,

$$\breve{S}_n = \breve{X}_1 + \breve{X}_2 + \cdots \breve{X}_n \leq X_1 + X_2 + \cdots X_n = S_n$$

Since $\breve{S}_n \leq S_n$, it follows for all $t > 0$ that if $S_n \leq t$ then also $\breve{S}_n \leq t$. This then means that if $N(t) \geq n$, then also $\breve{N}(t) \geq n$. Since this is true for all n, $\breve{N}(t) \geq N(t)$, i.e., the reduction of inter-renewal intervals causes an increase in the number of renewals.

c) Let M and $b < \infty$ such that $\mathbb{E}[\breve{X}] \geq M$ be fixed in what follows. Since $\breve{X} \leq b$, we see that $\mathbb{E}[\breve{X}] < \infty$, so we can apply Theorem 4.3.1, which asserts that

$$\Pr \omega : \lim_{t\to\infty} \frac{\breve{N}(t,\omega)}{t} = \frac{1}{\mathbb{E}[\breve{X}]} = 1$$

Let A denote the set of sample points above for which the above limit exists, i.e., for which $\lim_{t\to\infty} \breve{N}(t,\omega)/t = 1/\mathbb{E}[\breve{X}]$. We will show that, for each $\omega \in A$, $\lim_t N(t,\omega)/t \leq 1/2M$. We know that any for $\omega \in A$, $\lim_t \breve{N}(t,\omega)/t = 1/\mathbb{E}[\breve{X}]$. The definition of the limit of a real valued function states that for any $\epsilon > 0$, there is a $\tau(\epsilon)$ such that

$$\left| \frac{\breve{N}(t,\omega)}{t} - \frac{1}{\mathbb{E}[\breve{X}]} \right| \leq \epsilon \qquad \text{for all}\, t \geq \tau(\epsilon)$$

Note that $\tau(\epsilon)$ depends on b and ω as well as ϵ, so we denote it as $\tau(\epsilon, b, \omega)$. Using only one side of this inequality, $N(t,\omega)/t \leq \epsilon + 1/\mathbb{E}[\breve{X}]$ for all $t \geq \tau(\epsilon, b, \omega)$. Since we have seen that $N(t,\omega) \leq \breve{N}(t,\omega)$ and $\breve{X} \leq M$, we have

$$\frac{N(t,\omega)}{t} \leq \epsilon + \frac{1}{M} \qquad \text{for all}\, t \geq \tau(\epsilon, b, \omega)$$

Since ϵ is arbitrary, we can choose it as $1/M$, giving the desired inequality for all $\omega \in A$. Now for each choice of integer M, let $A(M)$ be the set of probability 1 above. The intersection of these sets also has probability 1, and each ω in all of these sets have

$\lim_t N(t,\omega)/t) = 0$. If you did this correctly, you should surely be proud of yourself!!!

Exercise 6:

a) In order to find $\mathbb{E}[N_s(t)]$, we need to use the iterative expectation formula and find $\mathbb{E}[N_s(t)|X_0 = j]$ first.

$$
\begin{aligned}
\mathbb{E}[N_s(t)] &= \mathbb{E}_{X_0}[\mathbb{E}[N_s(t)|X_0 = j]] \\
&= \sum_{j=1}^{M} \Pr\{X_0 = j\}\mathbb{E}[N_s(t)|X_0 = j] \\
&= \sum_{j=1}^{M} \pi_j \mathbb{E}[N_s(t)|X_0 = j]
\end{aligned}
$$

Knowing the first initial state, we can find the expected reward up to time t:

$$\mathbb{E}[N_s(t)|X_0 = j] = r_j + \sum_{i=1}^{M} P_{ji} r_i + \sum_{i=1}^{M} P_{ji}^2 r_i + \cdots + \sum_{i=1}^{M} P_{ji}^{t-1} r_i$$

Assuming that $\mathbb{E}[N_s(t)|X_0]$ is a vector in which the j-th element is $\mathbb{E}[N_s(t)|X_0 = j]$, we can write:

$$\mathbb{E}[N_s(t)|X_0] = \vec{r} + [P]\vec{r} + [P]^2\vec{r} + \cdots + [P]^{t-1}\vec{r}$$

$$
\begin{aligned}
\mathbb{E}[N_s(t)] &= \sum_{j=1}^{M} \pi_j \mathbb{E}[N_s(t)|X_0 = j] \\
&= \vec{\pi}\mathbb{E}[N_s(t)|X_0] \\
&= \vec{\pi}\left(\vec{r} + [P]\vec{r} + [P]^2\vec{r} + \cdots + [P]^{t-1}\vec{r}\right) \\
&= \vec{\pi}\vec{r} + \vec{\pi}[P]\vec{r} + \vec{\pi}[P]^2\vec{r} + \cdots + \vec{\pi}[P]^{t-1}\vec{r} \\
&= t\vec{\pi}\vec{r}
\end{aligned}
$$

The last equation is due to the fact that π is the steady state probability vector of the Markov chain and thus it is a left eigenvector of $[P]$ with eigenvalue $\lambda = 1$. Thus, $\vec{\pi}[P]^k = \vec{\pi}$.

Choosing the rewards as described in the problem where $r_1 = 1$ and for $j = \{2, \cdots M\}$ $r_j = 0$, we get: $\mathbb{E}[N_s(t)] = \pi_1 t$.

From the previous part, we would know that $\lim_{t\to\infty} \mathbb{E}[N_s(t)]/t = \lim_{t\to\infty} \pi_1 t/t = \pi_1$.

The difference between $N_s(t)$ and $N_1(t)$ is that the first process starts in steady state and the second starts in state 1. The second is a bona-fide renewal counting process and the first is what is called a delayed renewal counting process. After the first occurrence of state 1, in $N_s(t)$, the intervals between successive occurrences of state 1 are IID and the same as with $N_1(t)$. Thus the time to the nth renewal (starting in steady state) is $S_n = Y_1 + X_2 + X_3 + \cdots X_n$ where $X_2, \cdots X_n$ are IID and Y_1 is another rv. It is not hard to believe (Section 4.8 of the text makes it precise) that $\lim_n S_n/n = \bar{X}$ WP1 for the process starting in steady state, and $N_s(t)/t$ then converges WP1 to $1/\bar{X}$. From the above analysis of steady state, $\bar{X} = 1/\pi_1$.

b) The strong law for renewals say that if X_i is defined to be the i-th interarrival time of going from state 1 to itself, then with probability 1, $\lim_{t\to\infty} N_1(t)/t = 1/\bar{X}$. Thus, the expectation of the interarrival times of recurrence of state 1 is $1/\pi_1$