6.254 : Game Theory with Engineering Applications
Lecture 5: Existence of a Nash Equilibrium

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Outline

- Pricing-Congestion Game Example
- Existence of a Mixed Strategy Nash Equilibrium in Finite Games
- Existence in Games with Infinite Strategy Spaces

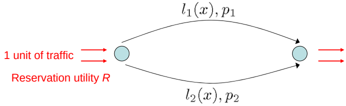
- Reading:
  - Fudenberg and Tirole, Chapter 1.

Introduction

- In this lecture, we study the question of existence of a Nash equilibrium in both games with finite and infinite pure strategy spaces.
- We start with an example, pricing-congestion game, where players have infinitely many pure strategies.
- We consider two instances of this game, one of which has a unique pure Nash equilibrium, and the other does not have any pure Nash equilibria.

Pricing-Congestion Game

- Consider a price competition model studied in [Acemoglu and Ozdaglar 07].



- Consider a parallel link network with I links. Assume that d units of flow is to be routed through this network. We assume that this flow is the aggregate flow of many *infinitesimal* users.
- Let $l_i(x_i)$ denote the latency function of link i, which represents the delay or congestion costs as a function of the total flow x_i on link i.
- Assume that the links are owned by independent providers. Provider i sets a price p_i per unit of flow on link i.
- The effective cost of using link i is $p_i + l_i(x_i)$.
- Users have a reservation utility equal to R, i.e., if $p_i + l_i(x_i) > R$, then no traffic will be routed on link i.

Example 1

- We consider an example with two links and latency functions $l_1(x_1) = 0$ and $l_2(x_2) = \frac{3x_2}{2}$. For simplicity, we assume that $R = 1$ and $d = 1$.
- Given the prices (p_1, p_2), we assume that the flow is allocated according to Wardrop equilibrium, i.e., the flows are routed along minimum effective cost paths and the effective cost cannot exceed the reservation utility.

Definition

A flow vector $x = [x_i]_{i=1,\ldots,I}$ *is a Wardrop equilibrium if* $\sum_{i=1}^{I} x_i \leq d$ *and*

$$p_i + l_i(x_i) = \min_j \{p_j + l_j(x_j)\}, \qquad \textit{for all } i \textit{ with } x_i > 0,$$

$$p_i + l_i(x_i) \leq R, \qquad \textit{for all } i \textit{ with } x_i > 0,$$

with $\sum_{i=1}^{I} x_i = d$ *if* $\min_j\{p_j + l_j(x_j)\} < R$.

Example 1 (Continued)

- We use the preceding characterization to determine the flow allocation on each link given prices $0 \leq p_1, p_2 \leq 1$:

$$x_2(p_1, p_2) = \begin{cases} \frac{2}{3}(p_1 - p_2), & p_1 \geq p_2, \\ 0, & \text{otherwise}, \end{cases}$$

  and $x_1(p_1, p_2) = 1 - x_2(p_1, p_2)$.

- The payoffs for the providers are given by:

$$u_1(p_1, p_2) = p_1 \times x_1(p_1, p_2)$$
$$u_2(p_1, p_2) = p_2 \times x_2(p_1, p_2)$$

- We find the pure strategy Nash equilibria of this game by characterizing the best response correspondences, $B_i(p_{-i})$ for each player.
  - The following analysis assumes that at the Nash equilibria (p_1, p_2) of the game, the corresponding Wardrop equilibria x satisfies $x_1 > 0$, $x_2 > 0$, and $x_1 + x_2 = 1$. For the proofs of these statements, see [Acemoglu and Ozdaglar 07].

Example 1 (Continued)

- In particular, for a given p_2, $B_1(p_2)$ is the optimal solution set of the following optimization problem

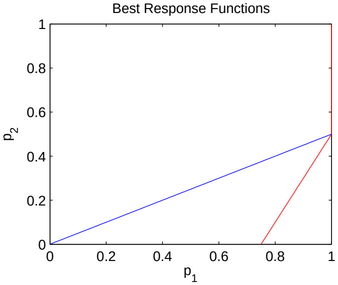
$$\begin{aligned} \text{maximize}_{\,0\leq p_1\leq 1,\ 0\leq x_1\leq 1} \quad & p_1 x_1 \\ \text{subject to} \quad & p_1 = p_2 + \frac{3}{2}(1 - x_1) \end{aligned}$$

- Solving the preceding optimization problem, we find that

$$B_1(p_2) = \min\left\{1, \frac{3}{4} + \frac{p_2}{2}\right\}.$$

Similarly, $B_2(p_1) = \frac{p_1}{2}$.

Example 1 (Continued)



- The figure illustrates the best response correspondences as a function of p_1 and p_2. The correspondences intersect at the unique point $(p_1, p_2) = (1, \frac{1}{2})$, which is the unique pure strategy equilibrium.

Example 2

- We next consider a similar example with latency functions given by

$$l_1(x) = 0, \qquad l_2(x) = \begin{cases} 0 & \text{if } 0 \leq x \leq 1/2 \\ \frac{x-1/2}{\epsilon} & x \geq 1/2, \end{cases}$$

  for some sufficiently small $\epsilon > 0$.

- The following list considers all candidate Nash equilibria (p_1, p_2) and profitable unilateral deviations for ϵ sufficiently small, thus establishing the nonexistence of a pure strategy Nash equilibrium:
  - $p_1 = p_2 = 0$: A small increase in the price of provider 1 will generate positive profits, thus provider 1 has an incentive to deviate.
  - $p_1 = p_2 > 0$: Let x be the corresponding flow allocation. If $x_1 = 1$, then provider 2 has an incentive to decrease its price. If $x_1 < 1$, then provider 1 has an incentive to decrease its price.
  - $0 \leq p_1 < p_2$: Player 1 has an incentive to increase its price since its flow allocation remains the same.
  - $0 \leq p_2 < p_1$: For ϵ sufficiently small, the profit function of player 2, given p_1, is strictly increasing as a function of p_2, showing that provider 2 has an incentive to increase its price.

Existence Results

- We start by analyzing existence of a Nash equilibrium in finite (strategic form) games, i.e., games with finite strategy sets.

Theorem

(Nash) *Every finite game has a mixed strategy Nash equilibrium.*

- Implication: matching pennies game necessarily has a mixed strategy equilibrium.
- Why is this important?
  - Without knowing the existence of an equilibrium, it is difficult (perhaps meaningless) to try to understand its properties.
  - Armed with this theorem, we also know that every finite game has an equilibrium, and thus we can simply try to locate the equilibria.

Approach

- Recall that a mixed strategy profile σ^* is a NE if

$$u_i(\sigma_i^*, \sigma_{-i}^*) \geq u_i(\sigma_i, \sigma_{-i}^*), \qquad \text{for all } \sigma_i \in \Sigma_i.$$

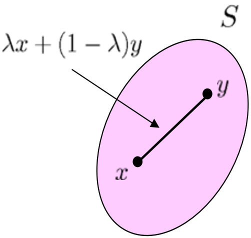
- In other words, σ^* is a NE if and only if $\sigma_i^* \in B_{-i}^*(\sigma_{-i}^*)$ for all i, where $B_{-i}^*(\sigma_{-i}^*)$ is the best response of player i, given that the other players' strategies are σ_{-i}^*.
- We define the correspondence $B : \Sigma \rightrightarrows \Sigma$ such that for all $\sigma \in \Sigma$, we have

$$B(\sigma) = [B_i(\sigma_{-i})]_{i \in \mathcal{I}} \tag{1}$$

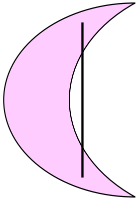
- The existence of a Nash equilibrium is then equivalent to the existence of a mixed strategy σ such that $\sigma \in B(\sigma)$: i.e., existence of a fixed point of the mapping B.
- We will establish existence of a Nash equilibrium in finite games using a fixed point theorem.

Definitions

- A set in a Euclidean space is compact if and only if it is bounded and closed.
- A set S is **convex** if for any $x, y \in S$ and any $\lambda \in [0, 1]$, $\lambda x + (1 - \lambda)y \in S$.



convex set



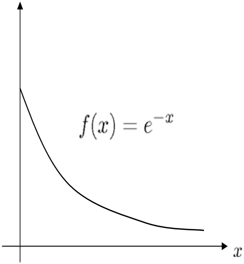
not a convex set

Weierstrass's Theorem

Theorem

(Weierstrass) *Let A be a nonempty compact subset of a finite dimensional Euclidean space and let $f : A \to \mathbb{R}$ be a continuous function. Then there exists an optimal solution to the optimization problem*

$$\begin{aligned} \textit{minimize} \quad & f(x) \\ \textit{subject to} \quad & x \in A. \end{aligned}$$



$\min_{x \geq 0} \; e^{-x} = 0$

There exists no optimal x that attains it

Kakutani's Fixed Point Theorem

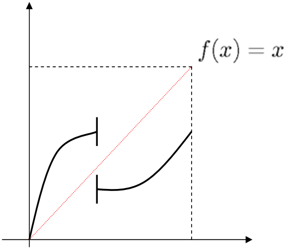
Theorem

(Kakutani) *Let A be a non-empty subset of a finite dimensional Euclidean space. Let $f : A \rightrightarrows A$ be a correspondence, with $x \in A \mapsto f(x) \subseteq A$, satisfying the following conditions:*

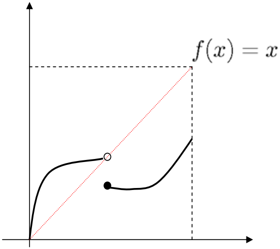
- *A is a compact and convex set.*
- *$f(x)$ is non-empty for all $x \in A$.*
- *$f(x)$ is a convex-valued correspondence: for all $x \in A$, $f(x)$ is a convex set.*
- *$f(x)$ has a closed graph: that is, if $\{x^n, y^n\} \to \{x, y\}$ with $y^n \in f(x^n)$, then $y \in f(x)$.*

Then, f has a fixed point, that is, there exists some $x \in A$, such that $x \in f(x)$.

Kakutani's Fixed Point Theorem—Graphical Illustration



$f(x)$ is not convex-valued



$f(x)$ does not have a closed graph

Proof of Nash's Theorem

- The idea is to apply Kakutani's theorem to the best response correspondence $B : \Sigma \rightrightarrows \Sigma$. We show that $B(\sigma)$ satisfies the conditions of Kakutani's theorem.

- Σ is compact, convex, and non-empty.
  - By definition
$$\Sigma = \prod_{i \in \mathcal{I}} \Sigma_i$$
    where each $\Sigma_i = \{x \mid \sum_j x_j = 1\}$ is a *simplex* of dimension $|S_i| - 1$, thus each Σ_i is closed and bounded, and thus compact. Their product set is also compact.

- $B(\sigma)$ is non-empty.
  - By definition,
$$B_i(\sigma_{-i}) = \arg\max_{x \in \Sigma_i} u_i(x, \sigma_{-i})$$
    where Σ_i is non-empty and compact, and u_i is linear in x. Hence, u_i is continuous, and by Weirstrass's theorem $B(\sigma)$ is non-empty.

Proof (continued)

3. $B(\sigma)$ is a convex-valued correspondence.
   - Equivalently, $B(\sigma) \subset \Sigma$ is convex if and only if $B_i(\sigma_{-i})$ is convex for all i. Let $\sigma'_i, \sigma''_i \in B_i(\sigma_{-i})$.
   - Then, for all $\lambda \in [0,1] \in B_i(\sigma_{-i})$, we have

$$u_i(\sigma'_i, \sigma_{-i}) \geq u_i(\tau_i, \sigma_{-i}) \qquad \text{for all } \tau_i \in \Sigma_i,$$

$$u_i(\sigma''_i, \sigma_{-i}) \geq u_i(\tau_i, \sigma_{-i}) \qquad \text{for all } \tau_i \in \Sigma_i.$$

   - The preceding relations imply that for all $\lambda \in [0,1]$, we have

$$\lambda u_i(\sigma'_i, \sigma_{-i}) + (1-\lambda) u_i(\sigma''_i, \sigma_{-i}) \geq u_i(\tau_i, \sigma_{-i}) \qquad \text{for all } \tau_i \in \Sigma_i.$$

   By the linearity of u_i,

$$u_i(\lambda \sigma'_i + (1-\lambda)\sigma''_i, \sigma_{-i}) \geq u_i(\tau_i, \sigma_{-i}) \qquad \text{for all } \tau_i \in \Sigma_i.$$

   Therefore, $\lambda \sigma'_i + (1-\lambda)\sigma''_i \in B_i(\sigma_{-i})$, showing that $B(\sigma)$ is convex-valued.

Proof (continued)

4. $B(\sigma)$ has a closed graph.
   - Suppose to obtain a contradiction, that $B(\sigma)$ does not have a closed graph.
   - Then, there exists a sequence $(\sigma^n, \hat{\sigma}^n) \to (\sigma, \hat{\sigma})$ with $\hat{\sigma}^n \in B(\sigma^n)$, but $\hat{\sigma} \notin B(\sigma)$, i.e., there exists some i such that $\hat{\sigma}_i \notin B_i(\sigma_{-i})$.
   - This implies that there exists some $\sigma_i' \in \Sigma_i$ and some $\epsilon > 0$ such that

$$u_i(\sigma_i', \sigma_{-i}) > u_i(\hat{\sigma}_i, \sigma_{-i}) + 3\epsilon.$$

   - By the continuity of u_i and the fact that $\sigma_{-i}^n \to \sigma_{-i}$, we have for sufficiently large n,

$$u_i(\sigma_i', \sigma_{-i}^n) \geq u_i(\sigma_i', \sigma_{-i}) - \epsilon.$$

Proof (continued)

- [step 4 continued] Combining the preceding two relations, we obtain

$$u_i(\sigma'_i, \sigma^n_{-i}) > u_i(\hat{\sigma}_i, \sigma_{-i}) + 2\epsilon \geq u_i(\hat{\sigma}^n_i, \sigma^n_{-i}) + \epsilon,$$

  where the second relation follows from the continuity of u_i. This contradicts the assumption that $\hat{\sigma}^n_i \in B_i(\sigma^n_{-i})$, and completes the proof.

- The existence of the fixed point then follows from Kakutani's theorem.
- If $\sigma^* \in B(\sigma^*)$, then by definition σ^* is a mixed strategy equilibrium.

Existence of Equilibria for Infinite Games

- A similar theorem to Nash's existence theorem applies for pure strategy existence in infinite games.

Theorem

(Debreu, Glicksberg, Fan) *Consider a strategic form game* $\langle \mathcal{I}, (S_i)_{i \in \mathcal{I}}, (u_i)_{i \in \mathcal{I}} \rangle$ *such that for each* $i \in \mathcal{I}$

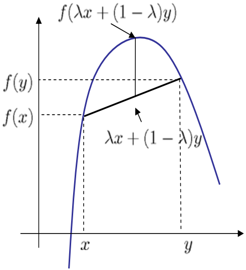
- S_i *is compact and convex;*
- $u_i(s_i, s_{-i})$ *is continuous in* s_{-i}*;*
- $u_i(s_i, s_{-i})$ *is continuous and concave in* s_i *[in fact quasi-concavity suffices].*

Then a pure strategy Nash equilibrium exists.

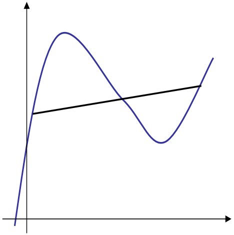
Definitions

- Suppose S is a convex set. Then a function $f : S \to \mathbb{R}$ is **concave** if for any $x, y \in S$ and any $\lambda \in [0, 1]$, we have

$$f(\lambda x + (1-\lambda)y) \geq \lambda f(x) + (1-\lambda) f(y).$$



concave function



not a concave function

Proof

- Now define the best response correspondence for player i, $B_i : S_{-i} \rightrightarrows S_i$,

$$B_i\left(s_{-i}\right) = \left\{s_i' \in S_i \mid u_i(s_i', s_{-i}) \geq u_i(s_i, s_{-i}) \text{ for all } s_i \in S_i\right\}.$$

  Thus restriction to pure strategies.

- Define the set of best response correspondences as

$$B\left(s\right) = \left[B_i\left(s_{-i}\right)\right]_{i \in \mathcal{I}}.$$

  and

$$B : S \rightrightarrows S.$$

Proof (continued)

- We will again apply Kakutani's theorem to the best response correspondence $B : S \rightrightarrows S$ by showing that $B(s)$ satisfies the conditions of Kakutani's theorem.

- S is compact, convex, and non-empty.
  - By definition
$$S = \prod_{i \in \mathcal{I}} S_i$$
since each S_i is compact [convex, nonempty] and finite product of compact [convex, nonempty] sets is compact [convex, nonempty].

- $B(s)$ is non-empty.
  - By definition,
$$B_i(s_{-i}) = \arg\max_{s \in S_i} u_i(s, s_{-i})$$
where S_i is non-empty and compact, and u_i is continuous in s by assumption. Then by Weirstrass's theorem $B(s)$ is non-empty.

Proof (continued)

3. $B(s)$ is a convex-valued correspondence.
   - This follows from the fact that $u_i(s_i, s_{-i})$ is concave [or quasi-concave] in s_i. Suppose not, then there exists some i and some $s_{-i} \in S_{-i}$ such that $B_i(s_{-i}) \in \arg\max_{s \in S_i} u_i(s, s_{-i})$ is not convex.
   - This implies that there exists $s_i', s_i'' \in S_i$ such that $s_i', s_i'' \in B_i(s_{-i})$ and $\lambda s_i' + (1-\lambda)s_i'' \notin B_i(s_{-i})$. In other words,

     $$\lambda u_i(s_i', s_{-i}) + (1-\lambda) u_i(s_i'', s_{-i}) > u_i(\lambda s_i' + (1-\lambda) s_i'', s_{-i}).$$

     But this violates the concavity of $u_i(s_i, s_{-i})$ in s_i [recall that for a concave function $f(\lambda x + (1-\lambda)y) \geq \lambda f(x) + (1-\lambda) f(y)$].
   - Therefore $B(s)$ is convex-valued.
4. The proof that $B(s)$ has a closed graph is identical to the previous proof.

Remarks

- Nash's theorem is a special case of this theorem: Strategy spaces are simplices and utilities are linear in (mixed) strategies, hence they are concave functions of (mixed) strategies.

Continuity properties of the "Nash equilibrium set":

- Consider strategic form games with finite pure strategy sets S_i and utilities $u_i(s, \lambda)$, where u_i is a continuous function of λ.
- Let $G(\lambda) = \langle \mathcal{I}, (S_i), (u_i(s, \lambda)) \rangle$ and let $E(\lambda)$ denote the Nash correspondence that associates with each λ, the set of (mixed) Nash equilibria of $G(\lambda)$.

Proposition

Assume that $\lambda \in \Lambda$, where Λ is a compact set. Then $E(\lambda)$ has a closed graph.

- Proof similar to the proof of closedness of $B(\sigma)$ in Nash's theorem.
- This does not imply continuity of the Nash equilibrium set $E(\lambda)$!!

Existence of Nash Equilibria

- Can we relax (quasi)concavity?
- Example: Consider the game where two players pick a location $s_1, s_2 \in \mathbb{R}^2$ on the circle. The payoffs are

$$u_1(s_1, s_2) = -u_2(s_1, s_2) = d(s_1, s_2),$$

  where $d(s_1, s_2)$ denotes the Euclidean distance between $s_1, s_2 \in \mathbb{R}^2$.
- No pure Nash equilibrium.
- However, it can be shown that the strategy profile where both mix uniformly on the circle is a mixed Nash equilibrium.

A More Powerful Theorem

Theorem

(Glicksberg) *Consider a strategic form game* $\langle \mathcal{I}, (S_i)_{i\in\mathcal{I}}, (u_i)_{i\in\mathcal{I}} \rangle$ *such that for each* $i \in \mathcal{I}$

- S_i *is a nonempty and compact metric space;*
- $u_i(s_i, s_{-i})$ *is continuous in* s.

Then a mixed strategy Nash equilibrium exists.

- With continuous strategy spaces, space of mixed strategies infinite dimensional!
- We will prove this theorem in the next lecture.