

15.081J/6.251J Introduction to Mathematical
Programming

Lecture 24: Discrete Optimization

1 Outline

SLIDE 1

- Modeling with integer variables
- What is a good formulation?
- Theme: The Power of Formulations

2 Integer Programming

2.1 Mixed IP

SLIDE 2

$$\begin{aligned} \text{(MIP)} \quad & \max \quad \mathbf{c}'\mathbf{x} + \mathbf{h}'\mathbf{y} \\ \text{s.t.} \quad & \mathbf{Ax} + \mathbf{By} \leq \mathbf{b} \\ & \mathbf{x} \in \mathbb{Z}_+^n (\mathbf{x} \geq 0, \mathbf{x} \text{ integer}) \\ & \mathbf{y} \in \mathbb{R}_+^n (\mathbf{y} \geq 0) \end{aligned}$$

2.2 Pure IP

SLIDE 3

$$\begin{aligned} \text{(IP)} \quad & \max \quad \mathbf{c}'\mathbf{x} \\ \text{s.t.} \quad & \mathbf{Ax} \leq \mathbf{b} \\ & \mathbf{x} \in \mathbb{Z}_+^n \end{aligned}$$

Important special case: Binary IP

$$\begin{aligned} \text{(BIP)} \quad & \max \quad \mathbf{c}'\mathbf{x} \\ \text{s.t.} \quad & \mathbf{Ax} \leq \mathbf{b} \\ & \mathbf{x} \in \{0, 1\}^n \end{aligned}$$

2.3 LP

SLIDE 4

$$\begin{aligned} \text{(LP)} \quad & \max \quad \mathbf{c}'\mathbf{x} \\ \text{s.t.} \quad & \mathbf{By} \leq \mathbf{b} \\ & \mathbf{y} \in \mathbb{R}_+^n \end{aligned}$$

3 Modeling with Binary Variables

3.1 Binary Choice

SLIDE 5

$$x \in \begin{cases} 1, & \text{if event occurs} \\ 0, & \text{otherwise} \end{cases}$$

Example 1: IP formulation of the knapsack problem

n : projects, total budget b

a_j : cost of project j

c_j : value of project j

SLIDE 6

$$x_j = \begin{cases} 1, & \text{if project } j \text{ is selected.} \\ 0, & \text{otherwise.} \end{cases}$$

$$\begin{aligned} \max \quad & \sum_{j=1}^n c_j x_j \\ \text{s.t.} \quad & \sum a_j x_j \leq b \\ & x_j \in \{0, 1\} \end{aligned}$$

3.2 Modeling relations

SLIDE 7

- At most one event occurs

$$\sum_j x_j \leq 1$$

- Neither or both events occur

$$x_2 - x_1 = 0$$

- If one event occurs then, another occurs

$$0 \leq x_2 \leq x_1$$

- If $x = 0$, then $y = 0$; if $x = 1$, then y is unconstrained

$$0 \leq y \leq Ux, \quad x \in \{0, 1\}$$

3.3 The assignment problem

SLIDE 8

n people
 m jobs
 c_{ij} : cost of assigning person j to job i .
 $x_{ij} = \begin{cases} 1 & \text{person } j \text{ is assigned to job } i \\ 0 & \end{cases}$
 $\min \sum_{j=1}^n c_{ij} x_{ij}$
 $\text{s.t.} \sum_{j=1}^n x_{ij} = 1$ each job is assigned
 $\sum_{i=1}^m x_{ij} \leq 1$ each person can do at most one job.
 $x_{ij} \in \{0, 1\}$

3.4 Multiple optimal solutions

SLIDE 9

- Generate all optimal solutions to a BOP.

$$\begin{aligned} \max \quad & \mathbf{c}'\mathbf{x} \\ \text{s.t.} \quad & \mathbf{A}\mathbf{x} \leq \mathbf{b} \\ & \mathbf{x} \in \{0, 1\}^n \end{aligned}$$

- Generate third best?
- Extensions to MIO?

3.5 Nonconvex functions

SLIDE 10

- How to model $\min c(\mathbf{x})$, where $c(\mathbf{x})$ is piecewise linear but not convex?

4 What is a good formulation?

4.1 Facility Location

SLIDE 11

- Data

$N = \{1 \dots n\}$ potential facility locations

$I = \{1 \dots m\}$ set of clients

c_j : cost of facility placed at j

h_{ij} : cost of satisfying client i from facility j .

- Decision variables

$$x_{ij} = \begin{cases} 1, & \text{a facility is placed at location } j \\ 0, & \text{otherwise} \end{cases}$$
$$y_{ij} = \begin{cases} \text{fraction of demand of client } i \\ \text{satisfied by facility } j. \end{cases}$$

SLIDE 12

$$IZ_1 = \min \sum_{j=1}^n c_j x_j + \sum_{i=1}^m \sum_{j=1}^n h_{ij} y_{ij}$$
$$\text{s.t. } \sum_{j=1}^n y_{ij} = 1$$
$$y_{ij} \leq x_j$$
$$x_j \in \{0, 1\}, 0 \leq y_{ij} \leq 1.$$

SLIDE 13

Consider an alternative formulation.

$$IZ_2 = \min \sum_{j=1}^n c_j x_j + \sum_{i=1}^m \sum_{j=1}^n h_{ij} y_{ij}$$
$$\text{s.t. } \sum_{j=1}^n y_{ij} = 1$$
$$\sum_{i=1}^m y_{ij} \leq m \cdot x_j$$
$$x_j \in \{0, 1\}, 0 \leq y_{ij} \leq 1.$$

Are both valid?

Which one is preferable?

4.2 Observations

SLIDE 14

- $IZ_1 = IZ_2$, since the integer points both formulations define are the same.

-

$$P_1 = \{(\mathbf{x}, \mathbf{y}) : \sum_{j=1}^n y_{ij} = 1, y_{ij} \leq x_j, \begin{cases} 0 \leq x_j \leq 1 \\ 0 \leq y_{ij} \leq 1 \end{cases} \}$$

$$P_2 = \left\{ (\mathbf{x}, \mathbf{y}) : \sum_{j=1}^n y_{ij} = 1, \sum_{i=1}^m y_{ij} \leq m \cdot x_j, \right. \\ \left. \begin{array}{l} 0 \leq x_j \leq 1 \\ 0 \leq y_{ij} \leq 1 \end{array} \right\}$$

SLIDE 15

- Let

$$Z_1 = \min_{(\mathbf{x}, \mathbf{y}) \in P_1} \mathbf{c}\mathbf{x} + \mathbf{h}\mathbf{y}, \quad Z_2 = \min_{(\mathbf{x}, \mathbf{y}) \in P_2} \mathbf{c}\mathbf{x} + \mathbf{h}\mathbf{y}$$

- $Z_2 \leq Z_1 \leq IZ_1 = IZ_2$

4.3 Implications

SLIDE 16

- Finding $IZ_1 (= IZ_2)$ is difficult.
- Solving to find Z_1, Z_2 is an LP. Since Z_1 is closer to IZ_1 several methods (branch and bound) would work better (actually much better).
- Suppose that if we solve $\min \mathbf{c}\mathbf{x} + \mathbf{h}\mathbf{y}, (\mathbf{x}, \mathbf{y}) \in P_1$ we find an integral solution. Have we solved the facility location problem?

SLIDE 17

- Formulation 1 is better than Formulation 2. (Despite the fact that 1 has a larger number of constraints than 2.)
- What is then the criterion?

4.4 Ideal Formulations

SLIDE 18

- Let P be an LP relaxation for a problem
- Let

$$H = \{(\mathbf{x}, \mathbf{y}) : \mathbf{x} \in \{0, 1\}^n\} \cap P$$

- Consider Convex Hull (H)

$$= \left\{ \mathbf{x} : \mathbf{x} = \sum_i \lambda_i \mathbf{x}^i, \sum_i \lambda_i = 1, \lambda_i \geq 0, \mathbf{x}^i \in H \right\}$$

SLIDE 19

- The extreme points of $CH(H)$ have $\{0, 1\}$ coordinates.
- So, if we know $CH(H)$ explicitly, then by solving $\min \mathbf{c}\mathbf{x} + \mathbf{h}\mathbf{y}, (\mathbf{x}, \mathbf{y}) \in CH(H)$ we solve the problem.
- Message: Quality of formulation is judged by closeness to $CH(H)$.

$$CH(H) \subseteq P_1 \subseteq P_2$$

5 Minimum Spanning Tree (MST)

SLIDE 20

- How do telephone companies bill you?
- It used to be that rate/minute: Boston \rightarrow LA proportional to distance in MST
- Other applications: Telecommunications, Transportation (good lower bound for TSP)

SLIDE 21

- Given a graph $G = (V, E)$ undirected and Costs $c_e, e \in E$.
- Find a tree of minimum cost spanning all the nodes.

- Decision variables $x_e = \begin{cases} 1, & \text{if edge } e \text{ is included in the tree} \\ 0, & \text{otherwise} \end{cases}$

SLIDE 22

- The tree should be connected. How can you model this requirement?
- Let S be a set of vertices. Then S and $V \setminus S$ should be connected

- Let $\delta(S) = \{e = (i, j) \in E : \begin{matrix} i \in S \\ j \in V \setminus S \end{matrix}\}$

- Then,

$$\sum_{e \in \delta(S)} x_e \geq 1$$

- What is the number of edges in a tree?
- Then, $\sum_{e \in E} x_e = n - 1$

5.1 Formulation

SLIDE 23

$$IZ_{MST} = \min \sum_{e \in E} c_e x_e$$

$$H \begin{cases} \sum_{e \in \delta(S)} x_e \geq 1 & \forall S \subseteq V, S \neq \emptyset, V \\ \sum_{e \in E} x_e = n - 1 \\ x_e \in \{0, 1\}. \end{cases}$$

Is this a good formulation?

SLIDE 24

$$P_{cut} = \{ \mathbf{x} \in R^{|E|} : 0 \leq \mathbf{x} \leq \mathbf{e},$$

$$\sum_{e \in E} x_e = n - 1$$

$$\sum_{e \in \delta(S)} x_e \geq 1 \forall S \subseteq V, S \neq \emptyset, V \}$$

Is P_{cut} the $CH(H)$?

5.2 What is $CH(H)$?

SLIDE 25

Let

$$P_{sub} = \{x \in R^{|E|} : \sum_{e \in E} x_e = n - 1$$

$$\sum_{e \in E(S)} x_e \leq |S| - 1 \forall S \subseteq V, S \neq \emptyset, V\}$$

$$E(S) = \left\{ e = (i, j) : \begin{array}{l} i \in S \\ j \in S \end{array} \right\}$$

Why is this a valid IP formulation?

SLIDE 26

- Theorem: $P_{sub} = CH(H)$.
- $\Rightarrow P_{sub}$ is the best possible formulation.
- MESSAGE: Good formulations can have an exponential number of constraints.

6 The Traveling Salesman Problem

SLIDE 27

Given $G = (V, E)$ an undirected graph. $V = \{1, \dots, n\}$, costs $c_e \forall e \in E$. Find a tour that minimizes total length.

6.1 Formulation I

SLIDE 28

$$x_e = \begin{cases} 1, & \text{if edge } e \text{ is included in the tour.} \\ 0, & \text{otherwise.} \end{cases}$$

$$\begin{aligned} \min & \sum_{e \in E} c_e x_e \\ \text{s.t.} & \sum_{e \in \delta(S)} x_e \geq 2, \quad S \subseteq E \\ & \sum_{e \in \delta(i)} x_e = 2, \quad i \in V \\ & x_e \in \{0, 1\} \end{aligned}$$

6.2 Formulation II

SLIDE 29

$$\begin{aligned} \min & \sum_{e \in E} c_e x_e \\ \text{s.t.} & \sum_{e \in E(S)} x_e \leq |S| - 1, \quad S \subseteq E \\ & \sum_{e \in \delta(i)} x_e = 2, \quad i \in V \\ & x_e \in \{0, 1\} \end{aligned}$$

SLIDE 30

$$\begin{aligned}
P_{cut}^{TSP} &= \{x \in R^{|E|}; \sum_{e \in \delta(S)} x_e \geq 2, \sum_{e \in \delta(i)} x_e = 2 \\
&\quad 0 \leq x_e \leq 1\} \\
P_{sub}^{TSP} &= \{x \in R^{|E|}; \sum_{e \in \delta(i)} x_e = 2 \\
&\quad \sum_{e \in \delta(S)} x_e \leq |S| - 1 \\
&\quad 0 \leq x_e \leq 1\}
\end{aligned}$$

SLIDE 31

- Theorem: $P_{cut}^{TSP} = P_{sub}^{TSP} \not\subseteq CH(H)$
- Nobody knows $CH(H)$ for the TSP

7 Minimum Matching

SLIDE 32

- Given $G = (V, E)$; c_e costs on $e \in E$. Find a matching of minimum cost.
- Formulation:

$$\begin{aligned}
\min \quad & \sum c_e x_e \\
\text{s.t.} \quad & \sum_{e \in \delta(i)} x_e = 1, \quad i \in V \\
& x_e \in \{0, 1\}
\end{aligned}$$

- Is the LP relaxation $CH(H)$?

SLIDE 33

Let

$$\begin{aligned}
P_{MAT} &= \{x \in R^{|E|} : \sum_{e \in \delta(i)} x_e = 1 \\
&\quad \sum_{e \in \delta(S)} x_e \geq 1 \quad |S| = 2k + 1, S \neq \emptyset \\
&\quad x_e \geq 0\}
\end{aligned}$$

Theorem: $P_{MAT} = CH(H)$

8 Observations

SLIDE 34

- For MST, Matching there are efficient algorithms. $CH(H)$ is known.
- For TSP $\not\exists$ efficient algorithm. TSP is an $NP - hard$ problem. $CH(H)$ is not known.
- Conjecture: The convex hull of problems that are polynomially solvable are explicitly known.

9 Summary

SLIDE 35

1. An IP formulation is better than another one if the polyhedra of their LP relaxations are closer to the convex hull of the IP.
2. A good formulation can have an exponential number of constraints.
3. Conjecture: Formulations characterize the complexity of problems. If a problem is solvable in polynomial time, then the convex hull of solutions is known.

MIT OpenCourseWare
<http://ocw.mit.edu>

6.251J / 15.081J Introduction to Mathematical Programming
Fall 2009

For information about citing these materials or our Terms of Use, visit: <http://ocw.mit.edu/terms>.