15.081J/6.251J Introduction to Mathematical Programming

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1 Structure of Class

SLIDE 1

- Formulations: Lec. 1
- Geometry: Lec. 2-4
- Simplex Method: Lec. 5-8
- Duality Theory: Lec. 9-11
- Sensitivity Analysis: Lec. 12
- Robust Optimization: Lec. 13
- Large scale optimization: Lec. 14-15
- Nctmork Flows: Lec. 16-17
- The Ellipsoid method: Lec. 18-19
- Interior point methods: Lec. 20-21
- Semidefinite optimization: Lec. 22
- Discrete Optimization: Lec. 24-25

2 Requirements

SLIDE 2

- Homcmorkn: 30%
- Midterm Exam: 30%
- Final Exam: 40%
- Important tie braker: contributions to class

Use of CPLEX for solving optimization problems

3 Lecture Outline

SLIDE 3

- History of Optimization
- Where LOPs Arise?
- Examples of Formulations

4 History of Optimization

SLIDE 4

Fermat, 1638: Newton, 1670

$$\min f(x) \qquad x\text{: scalar}$$

$$\frac{df(x)}{dx} = 0$$

Euler, 1755

$$\min f(x_1, \ldots, x_n)$$

$$\nabla f(\boldsymbol{x}) = 0$$

Lagrange, 1707

$$\min \quad f(x_1, \ldots, x_n)$$

$$\text{s.t.} \quad g_k(x_1, \ldots, x_n) = 0 \qquad k = 1, \ldots, m$$

Euler, Lagrange Problems in infinite dimensions, calculus of variations.

5 Nonlinear Optimization

5.1 The general problem

SLIDE 5

$$\begin{array}{ll} \min & f(x_1, \ldots, x_n) \\ \text{s.t.} & g_1(x_1, \ldots, x_n) \leq 0 \\ & \\ & g_m(x_1, \ldots, x_n) \leq 0. \end{array}$$

6 What is Linear Optimization?

6.1 Formulation

SLIDE 6

$$\begin{array}{ll} \text{minimize} & 3x_1 + x_2 \\ \text{subject to} & x_1 + 21.2 \geq 2 \\ & 2x_1 + x_2 \geq 3 \\ & x_1 \geq 0.1.2 \geq 0 \end{array}$$

$$\boldsymbol{c} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}, \quad \boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad \boldsymbol{b} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \quad \boldsymbol{A} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$$

$$\begin{array}{ll} \text{minimize} & \boldsymbol{c}'\boldsymbol{x} \\ \text{subject to} & \boldsymbol{A}\boldsymbol{x} \geq h \\ & \boldsymbol{x} \geq \boldsymbol{0} \end{array}$$

7 History of LO

7.1 The pre-algorithmic period

SLIDE 7

Fourier, 1826 Method for solving system of linear inequalities.

de la Vallée Poussin simplex-like method for objective function with absolute values.

Kantorovich, Koopmans, 1930s Formulations and solution method

von Neumann, 1928 game theory, duality.

Farkas, Minkowski, Carathéodory, 1870-1930 Foundations

7.2 The modern period

SLIDE 8

George Dantzig, 1047 Simplex method

1950s Applications.

1960s Large Scale Optimization.

1970s Complexity theory.

1979 The ellipsoid algorithm.

1980s Interior point algorithms.

1990s Semidefinite and conic optimization.

2000s Robust Optimization.

8 Where do LOPs Arise?

8.1 Wide Applicability

SLIDE 9

- Transportation

  Air traffic control, Crew scheduling,
  Movement of Truck Loads

- Telecommunications
- Manufacturing
- Medicine
- Engineering
- Typesetting (TEX, LATEX)

9 Transportation Problem

9.1 Data

SLIDE 10

- m plants. n warehouses
- s_i supply of ith plant, $i = 1 \dots m$
- d_j demand of jth warehouse, $j = 1 \dots n$
- c_{ij}: cost of transportation $i \to j$

9.2 Decision Variables

9.2.1 Formulation

SLIDE 11

x_{ij} = number of units to send $i \to j$

$$\begin{aligned}
\min \quad & \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij} \\
\text{s.t.} \quad & \sum_{i=1}^{m} x_{ij} = d_j \qquad j = 1 \dots n \\
& \sum_{j=1}^{n} x_{ij} = s_i \qquad i = 1 \dots m \\
& x_{ij} \geq 0
\end{aligned}$$

10 Sorting through LO

SLIDE 12

- Given n numbers $c_1, c_2, \dots, c_n$;
- The order statistic $c_{(1)}, c_{(2)}, \dots, c_{(n)}$: $c_{(1)} \leq c_{(2)} \leq \dots \leq c_{(n)}$;
- Use LO to find $\sum_{i=1}^{k} c_{(i)}$.

$$\begin{aligned}
\min \quad & \sum_{i=1}^{n} c_i x_i \\
\text{s.t.} \quad & \sum_{i=1}^{n} x_i = k \\
& 0 \leq x_i \leq 1 \qquad i = 1, \dots, n
\end{aligned}$$

11 Investment under taxation

SLIDE 13

- You have purchased s_i shares of stock i at price q_i, $i = 1, \dots, n$
- Current price of stock i is p_i

- You expect that the price of stock i one year from now will be r_i
- You pay a capital-gains tax at the rate of 30% on any capital gains at the time of the sale.
- You want to raise C amount of cash after taxes.
- You pay 1% in transaction costs
- Example: You sell 1,000 shares at $50 per share; you have bought them at $30 per share: Net cash is:

$$\$50 \times 1,000 - 0.30 \times (\$50 - 30) \times 1,000$$

$$-0.01 \times 50 \times 1,000 = \$43,500$$

11.1 Formulation

SLIDE 14

$$\begin{array}{ll} \max & \sum_{i=1}^{n} r_i(s_i - x_i) \\ \text{s.t.} & \sum_{i=1}^{n} p_i x_i - 0.30 \sum_{i=1}^{n} (p_i - q_i) x_i - 0.01 \sum_{i=1}^{n} p_i x_i \geq C \\ & 0 \leq x_i \leq s_i \end{array}$$

12 Investment Problem

SLIDE 15

- Five investment choices A, B, C, D, E
- A, C, and D are available in 1993.
- B is available in 1994.
- E is available in 1995.
- Cash earns 6% per year.
- $1,000,000 in 1993.

12.1 Cash Flowper Dollar Invested

SLIDE 16

	A	B	C	D	E
1993	-1.00	0	-1.00	-1.00	0
1994	+0.30	-1.00	+1.10	0	0
1995	+1.00	+0.30	0	0	-1.00
1996	0	+1.00	0	+1.75	+1.40
LIMIT	$500,000	NONE	$500,000	NONE	$750,000

12.2 Formulation

12.2.1 Decision Variables

SLIDE 17

- $A, \ldots E$: amount invested in \$ millions
- $Cash_t$: amount invested in cash in period t, $t = 1, 2, 3$

$$\begin{array}{ll} \max & 1.06Cash_3 + 1.00B + 1.75D + 1.40E \\ \text{s.t.} & A + C + D + Cash_1 \leq 1 \\ & Cash_2 + B \leq 0.3A + 1.1C + 1.06Cash_1 \\ & Cash_3 + 1.0E \leq 1.0A + 0.3B + 1.06Cash_2 \\ & A \leq 0.5, \quad C \leq 0.5, \quad E \leq 0.75 \\ & A, \ldots, E \geq 0 \end{array}$$

- Solution: $A = 0.5M$, $B = 0$, $C = 0$, $D = 0.5M$, $E = 0.659M$, $Cash_1 = 0$, $Cash_2 = .15M$, $Cash_3 = 0$; Objective: $1.7976M$

13 Manufacturing

13.1 Data

SLIDE 18

- n products, m raw materials
- c_j: profit of product j
- b_i: available units of material i.
- a_{ij}: # units of material i product j needs in order to be produced.

13.2 Formulation

13.2.1 Decision variables

SLIDE 19

x_j = amount of product j produced.

$$\begin{array}{ll} \max & \sum_{j=1}^{n} c_j x_j \\ \text{s.t.} & a_{11}x_1 + \cdots + a_{1n}x_n \leq b_1 \\ & \vdots \\ & a_{m1}x_1 + \cdots + a_{mn}x_n \leq b_m \\ & x_j \geq 0, \qquad j = 1 \ldots n \end{array}$$

14 Capacity Expansion

14.1 Data and Constraints

SLIDE 20

D_t: forecasted demand for electricity at year t

E_t: existing capacity (in oil) available at t

c_t: cost to produce 1MW using coal capacity

n_t: cost to produce 1MW using nuclear capacity

- No more than 20% nuclear
- Coal plants last 20 years
- Nuclear plants last 15 years

14.2 Decision Variables

SLIDE 21

x_t: amount of <u>coal</u> capacity brought on line in year t.

y_t: amount of <u>nuclear</u> capacity brought on line in year t.

w_t: total <u>coal</u> capacity in year t.

z_t: total <u>nuclear</u> capacity in year t.

14.3 Formulation

SLIDE 22

$$
\begin{array}{lll}
\min & \sum_{t=1}^{T} c_t x_t + n_t y_t & \\
\text{s.t.} & w_t = \sum_{s=\max(0,t-19)}^{t} x_s, & t = 1 \ldots T \\
& z_t = \sum_{s=\max(0,t-14)}^{t} y_s, & t = 1 \ldots T \\
& w_t + z_t + E_t \geq D_t & \\
& z_t \leq 0.2(w_t + z_t + E_t) & \\
& x_t, y_t, w_t, z_t \geq 0. &
\end{array}
$$

15 Scheduling

15.1 Decision variables

SLIDE 23

- Hospital wants to make weekly nightshift for its nurses
- D_j demand for nurses, $j = 1 \ldots 7$
- Every nurse works 5 days in a row

- Goal: hire minimum number of nurses

Decision Variables
x_j: # nurses starting their week on day **j**

15.2 Formulation

SLIDE 24

$$
\begin{array}{llllllllll}
\min & \sum_{j=1}^{7} x_7 & & & & & & & & \\
\text{s.t.} & x_1+ & & & x_4+ & x_5+ & x_6+ & x_7 & \geq & d_1 \\
 & x_1+ & x_2 & & & x_5+ & x_6+ & x_7 & \geq & d_2 \\
 & x_1+ & x_2+ & x_3 & & & x_6+ & x_7 & \geq & d_3 \\
 & x_1+ & x_2+ & x_3+ & x_4+ & & & x_7 & \geq & d_4 \\
 & x_1+ & x_2+ & x_3+ & x_4+ & x_5 & & & \geq & d_5 \\
 & x_j \geq 0 & x_2+ & x_3+ & x_4+ & x_5+ & x_6 & & \geq & d_6 \\
 & & & x_3+ & x_4+ & x_5+ & x_6+ & x_7 & \geq & d_7
\end{array}
$$

16 Revenue Management

16.1 The industry

SLIDE 25

- Deregulation in 1978
- Prior to Deregulation
  - Carriers only allowed to fly certain routes. Hence airlines such as Northwest, Eastern, Southwest, etc.
  - Fares determined by Civil Aeronautics Board (CAB) based on mileage and other costs (CAB no longer exists)

SLIDE 26

Post Deregulation

- anyone can fly, anywhere
- fares determined by carrier (and the market)

17 Revenue Management

17.1 Economics

SLIDE 27

- Huge sunk and fixed costs
- Very low variable costs per passenger ($10/passenger or less)
- Strong economically competitive environment
- Near-perfect information and negligible cost of information
- Highly perishable inventory
- ***Result:*** Multiple fares

18 Revenue Management

18.1 Data

SLIDE 28

- n origins. n destinations
- I hub
- 2 classes (for simplicity), (2-class. Ti-class
- Revenues r_{ij}^Q, r_{ij}^Y
- Capacities: C_{i0}, $i = 1 \ldots, n$; C_{0j}, $j = 1, \ldots n$
- Expected demands: D_{ij}^Q, D_{ij}^Y

18.2 LO Formulation

18.2.1 Decision Variables

SLIDE 29

- Q_{ij}: #of Q-class customers we accept from i to j
- Y_{ij}: # of Y-class customers we accept from i to j

$$\begin{aligned} \text{maximize} \quad & \sum r_{ij}^Q Q_{ij} + r_{ij}^Y Y_{ij} \\ \text{subject to} \quad & \sum_{j=0}^{n} (Q_{ij} + Y_{ij}) \le C_{i0} \\ & \sum_{i=0}^{n} (Q_{ij} + Y_{ij}) \le C_{0j} \\ & 0 \le Q_{ij} \le D_{ij}^Q, \quad 0 \le Y_{ij} \le D_{ij}^Y \end{aligned}$$

19 Revenue Management

19.1 Importance

SLIDE 30

Robert Crandall, former CEO of American Airlines:

We estimate that RM has generated $1.4 billion in incremental revenue for American Airlines in the last three years alone. This is not a one-time benefit. We expect RM to generate at least $500 million annually for the foreseeable future. As we continue to invest in the enhancement of DINAMO we expect to capture nn even larger revenue premium.

20 Messages

20.1 How to formulate?

SLIDE 31

1. Define your decision variables clearly.
2. Write constraints and objective function.
3. No systematic method available

What is a good LO formulation?
A formulation with a small number of variables and constraints, and the matrix A is sparse.

21 Nonlinear Optimization

21.1 The general problem

SLIDE 32

$$\begin{array}{ll} \min & f(x_1, \ldots, x_n) \\ \text{s.t.} & g_1(x_1, \ldots, x_n) \leq 0 \\ & \vdots \\ & g_m(x_1, \ldots, x_n) \leq 0. \end{array}$$

22 Convex functions

SLIDE 33

- $\mathbf{f} : S \longrightarrow R$
- For all $x_1, x_2 \in S$

$$f(\lambda x_1 + (1-\lambda)x_2) \leq \lambda f(x_1) + (1-\lambda) f(x_2)$$

- $f(\mathrm{z})$concave if $-\mathbf{f}(\boldsymbol{x})$ convex.

23 On the power of LO

23.1 LO formulation

SLIDE 34

$$\begin{array}{ll} \min & \mathrm{f}(\boldsymbol{x}) = \text{maxi}, \quad \boldsymbol{d_k}\, \boldsymbol{x} + c_k \Big) \\ s.t. & \mathrm{Az} \geq \boldsymbol{b} \end{array}$$

$$\begin{array}{lll} \min & z & \\ s.t. & \boldsymbol{Ax} \geq \boldsymbol{b} & \\ & \boldsymbol{d_k}'\boldsymbol{x} + c_k \leq z & \forall\ k \end{array}$$

24 On the **power** of LO

24.1 Problems with $|.|$

Slide 35

$$\begin{array}{rl} \min & \sum c_j |x_j| \\ \text{s.t.} & \boldsymbol{Ax} \geq \boldsymbol{b} \end{array}$$

Idea: $|x| = \max\{x, -x\}$

$$\begin{array}{rl} \min & \sum c_j z_j \\ \text{5.t.} & \boldsymbol{Ax} \geq \boldsymbol{b} \\ & x_j \leq z_j \\ & -x_j \leq z_j \end{array}$$

Message: Minimizing Piecewise linear convex function can be modelled by LO

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