Massachusetts Institute of Technology

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6.243j (Fall 2003): DYNAMICS OF NONLINEAR SYSTEMS

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Lecture 8: Local Behavior at Eqilibria[1]

This lecture presents results which describe local behavior of autonomous systems in terms of Taylor series expansions of system equations in a neigborhood of an equilibrium.

8.1 First order conditions

This section describes the relation between eigenvalues of a *Jacobian* $a'(\bar{x}_0)$ and behavior of ODE

$$\dot{x}(t) = a(x(t)) \tag{8.1}$$

or a difference equation

$$x(t+1) = a(x(t)) \tag{8.2}$$

in a neigborhood of equilibrium $\bar{x}_0$.

In the statements below, it is assumed that $a: X \mapsto \mathbf{R}^n$ is a continuous function defined on an open subset $X \subset \mathbf{R}^n$. It is further assumed that $\bar{x}_0 \in X$, and there exists an n-by-n matrix A such that

$$\frac{|a(\bar{x}_0+\delta) - a(\bar{x}_0) - A\delta|}{|\delta|} \to 0 \;\text{ as }\; |\delta| \to 0. \tag{8.3}$$

If derivatives da_k/dx_i of each component a_k of a with respect to each cpomponent x_i of x exist at $\bar{x}_0$, A is the matrix with coefficients da_k/dx_i, i.e. the *Jacobian* of the system. However, differentiability at a single point $\bar{x}_0$ does not guarantee that (8.3) holds. On the other hand, (8.3) follows from *continuous differentiability* of a in a neigborhood of $\bar{x}_0$.

[1] Version of October 3, 2003

Example 8.1 Function $a: \mathbf{R}^2 \mapsto \mathbf{R}^2$, defined by

$$a\left(\begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix}\right) = \frac{\bar{x}_1^2\bar{x}_2^2 - (\bar{x}_1^2 - \bar{x}_2^2)^2}{\bar{x}_1^2\bar{x}_2^2 + (\bar{x}_1^2 - \bar{x}_2^2)^2} \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix}$$

for $\bar{x} \neq 0$, and by $a(0) = 0$, is differentiable with respect to $\bar{x}_1$ and $\bar{x}_2$ at every point $\bar{x} \in \mathbf{R}^2$, and its Jacobian $a'(0) = A$ equals minus identity matrix. However, condition (8.3) is not satisfied (note that $a'(\bar{x})$ is not continuous at $\bar{x} = 0$).

8.1.1 The continuous time case

Let us call an equilibrium $\bar{x}_0$ of (8.1) *exponentially stable* if there exist positive real numbers ϵ, r, C such that every solution $x : [0, T] \mapsto X$ with $|x(0) - \bar{x}_0| < \epsilon$ satisfies

$$|x(t) - \bar{x}_0| \leq Ce^{-rt}|x(0) - \bar{x}_0| \quad \forall\ t \geq 0.$$

The following theorem can be attributed directly to Lyapunov.

Theorem 8.1 *Assume that $a(\bar{x}_0) = 0$ and condition (8.3) is satisfied. Then*

(a) if $A = a'(\bar{x}_0)$ is a Hurwitz matrix (i.e. if all eigenvalues of A have negative real part) then $\bar{x}_0$ is a (locally) exponentially stable equilibrium of (8.1);

(b) if $A = a'(\bar{x}_0)$ has an eigenvalue with a non-negative real part then $\bar{x}_0$ is not *an exponentially stable equilibrium of (8.1);*

(c) if $A = a'(\bar{x}_0)$ has an eigenvalue with a positive real part then $\bar{x}_0$ is not *a stable equilibrium of (8.1).*

Note that Theorem 8.1 does not cover all possible cases: if A is not a Hurwitz matrix and does not have eigenvalues with positive real part then the statement says very little, and for a good reason: the equilibrium may turn out to be asymptotically stable or unstable. Note also that the equilibrium $\bar{x} = 0$ from Example 8.1 (where a is differentiable but does not satisfy (8.3)) is not stable, despite the fact that $A = -I$ has all eigenvalues at -1.

Example 8.2 The equilibrium $\bar{x} = 0$ of the ODE

$$\dot{x}(t) = \alpha x(t) + \beta x(t)^3$$

is asympotically stable when $\alpha < 0$ (this is due to Theorem 8.1), but also when $\alpha = 0$ and $\beta < 0$. The equilibrium is not stable when $\alpha > 0$ (due to Theorem 8.1), but also when $\alpha = 0$ and $\beta > 0$. In addition, the equilibrium is stable but not asymptotically stable when $\alpha = \beta = 0$.

8.1.2 Proof of Theorem 8.1

The proof of (a) can be viewed as an excercise in "storage function construction" outlined in the previous lecture. Indeed, assuming, for simplicity, that $\bar{x}_0 = 0$, (8.1) can be re-written as

$$\dot{x}(t) = Ax(t) + w(t), \quad w(t) = a(x(t)) - Ax(t).$$

Here the linear part has standard storage functions

$$V_{LTI}(\bar{x}) = \bar{x}'P\bar{x}, \quad P = P'$$

with supply rates

$$\sigma_{LTI}(\bar{x}, \bar{w}) = 2\bar{x}'P(A\bar{x} + \bar{w}).$$

In addition, due to (8.3), for every $\delta > 0$ there exists $\epsilon > 0$ such that the nonlinear component $w(t)$ satisfies the sector constraint

$$\sigma_{NL}(x(t), w(t)) = \delta|x(t)|^2 - |w(t)|^2 \geq 0,$$

as long as $|x(t)| < \epsilon$. Since A is a Hurwitx matrix, $P = P'$ can be chosen positive definite and such that

$$PA + A'P = -I.$$

Then

$$\begin{aligned} \sigma(\bar{x}, \bar{w}) &= \sigma_{LTI}(\bar{x}, \bar{w}) + \tau\sigma_{NL}(\bar{x}, \bar{w}) \\ &= (\tau\delta - 1)|\bar{x}|^2 + 2\tau\bar{x}'P\bar{w} - \tau|\bar{w}|^2 \leq (\tau\delta - 1)|\bar{x}|^2 - 2\|P\| \cdot |\bar{x}| \cdot |\bar{w}| - \tau|\bar{w}|^2, \end{aligned}$$

where $\|P\|$ is the largest singular value of P, is a supply rate for the storage function $V = V_{LTI}$ for every constant $\tau \geq 0$. When $\tau = 16\|P\|$ and $\delta = 0.25/\tau$, we have

$$\sigma(\bar{x}, \bar{w}) \leq -0.5|\bar{x}|^2,$$

which proves that, for $|x(t)| < \epsilon$, the inequality

$$V(x(t)) \leq -0.5|x(t)|^2 \leq -\frac{1}{2\|P^{-1}\|}V(x(t)).$$

Hence

$$V(x(t)) \leq e^{-dt}V(x(0)) \quad \forall\ t \geq 0,$$

where $d = 1/2\|P^{-1}\|$, as long as $|x(t)| < \epsilon$. Since

$$\|P\| \cdot |x(t)| \geq V(x(t)) \geq \|P^{-1}\|^{-1} \cdot |x(t)|,$$

this implies (a).

The proofs of (b) and (c) are more involved, based on showing that solutions which start at $\bar{x}_0 + \delta v$, where v is an eigenvector of A corresponding to an eigenvalue with a non-negative (strictly positive) real part, cannot converge to $\bar{x}_0$ quickly enough (respectively, diverge from $\bar{x}_0$).

To prove (b), take a real number $d \in (0, r/2)$ such that no two eigenvalues of A sum up to $-2d$. Then $P = P'$ be the unique solution of the Lyapunov equation

$$P(A + dI) + (A' + dI)P = -I.$$

Note that P is non-singular: otherwise, if $Pv = 0$ for some $v \neq 0$, it follows that

$$-|v|^2 = v'(P(A + dI) + (A' + dI)P)v = (Pv)'(A + dI)v + v'(A' + dI)(Pv) = 0.$$

In addition, $P = P'$ is not positive semidefinite: since, by assumption, $A + dI$ has an eigenvector $u \neq 0$ which corresponds to an eigenvalue λ with a positive real part, we have

$$-|u|^2 = -2\mathrm{Re}(\lambda)u'Pu,$$

hence $u'Pu < 0$.

Let $\epsilon > 0$ be small enough so that

$$2\bar{x}'Pw \leq 0.5|\bar{x}|^2 \quad \text{for} \quad |w| \leq \epsilon|\bar{x}|.$$

By assumption, there exists $\delta > 0$ such that

$$|a(\bar{x}) - A\bar{x}| \leq \epsilon|\bar{x}| \quad \text{for} \quad |\bar{x}| \leq \delta.$$

Then

$$\frac{d}{dt}(e^{2dt}x(t)'Px(t)) = e^{2dt}(2dx(t)'Px(t) + 2x(t)'PAx(t) + 2x(t)'P(a(x(t)) - Ax(t)))$$
$$\leq -0.5e^{2dt}|x(t)|^2$$

as long as $x(t)$ is a solution of (8.1) and $|x(t)| \leq \delta$. In particular, this means that if $x(0)'Px(0) \leq -R < 0$ and $|x(0)| \leq \delta$ then $e^{2dt}x(t)'Px(t) \leq -R$ for as long as $|x(t)| \leq \delta$, which contradicts exponential stability with rate $r > 2d$.

The proof of (c) is similar to that of (a).

8.1.3 The discrete time case

The results for the discrete time case are similar to Theorem 8.1, with the real parts of the eigenvalues being replaced by the difference between their absolute values and 1.

Let us call an equilibrium $\bar{x}_0$ of (8.2) *exponentially stable* if there exist positive real numbers ϵ, r, C such that every solution $x : [0, T] \mapsto X$ with $|x(0) - \bar{x}_0| < \epsilon$ satisfies

$$|x(t) - \bar{x}_0| \leq Ce^{-rt}|x(0) - \bar{x}_0| \quad \forall\, t = 0, 1, 2, \ldots.$$

Theorem 8.2 *Assume that $a(\bar{x}_0) = 0$ and condition (8.3) is satisfied. Then*

(a) *if $A = a'(\bar{x}_0)$ is a Schur matrix (i.e. if all eigenvalues of A have absolute value less than one) then $\bar{x}_0$ is a (locally) exponentially stable equilibrium of (8.2);*

(b) *if $A = a'(\bar{x}_0)$ has an eigenvalue with absolute value greater than 1 then $\bar{x}_0$ is* not *an exponentially stable equilibrium of (8.2);*

(c) *if $A = a'(\bar{x}_0)$ has an eigenvalue with absolute value strictly larger than 1 then $\bar{x}_0$ is* not *a stable equilibrium of (8.2).*

8.2 Higher order conditions

When the Jacobian $A = a'(\bar{x}_0)$ of (8.1) evaluated at the equilibrium $\bar{x}_0$ has no eigenvalues with positive real part, but has some eigenvalues on the imaginary axis, local stability analysis becomes much more complicated. Based on the proof of Theorem 8.1, it is natural to expect that system states corresponding to strictly stable eigenvalues will behave in a predictably stable fashion, and hence the behavior of system states corresponding to the eigenvalues on the imaginary axis will determine local stability or instability of the equilibrium.

8.2.1 A Center Manifold Theorem

In this subsection we assume for simplicity that $\bar{x}_0 = 0$ is the studied equilibrium of (8.1), i.e. $a(0) = 0$. Assume also that a is k times continuously differentiable in a neigborhood of $\bar{x}_0 = 0$, where $k \geq 1$, and that $A = a'(0)$ has no eigenvalues with positive real part, but has eigenvalues on the imaginary axis, as well as in the open left half plane $\mathrm{Re}(s) < 0$. Then a linear change of coordinates brings A into a block-diagonal form

$$A = \begin{bmatrix} A_c & 0 \\ 0 & A_s \end{bmatrix},$$

where A_s is a Hurwitz matrix, and all eigenvalues of A_c have zero real part.

Theorem 8.3 *Let $a : \mathbf{R}^n \mapsto \mathbf{R}^n$ be $k \geq 2$ times continuously differentiable in a neigborhood of $\bar{x}_0 = 0$. Assume that $a(0) = 0$ and*

$$a'(0) = A = \begin{bmatrix} A_c & 0 \\ 0 & A_s \end{bmatrix},$$

where A_s is a Hurwitz p-by-p matrix, and all eigenvalues of the q-by-q matrix A_c have zero real part. Then

(a) *there exists $\epsilon > 0$ and a function $h : \mathbf{R}^q \mapsto \mathbf{R}^p$, $k-1$ times continuously differentiable in a neigborhood of the origin, such that $h(0) = 0$, $h'(0) = 0$, and every solution $x(t) = [x_c(t); x_s(t)]$ of (8.1) with $x_s(0) = h(x_c(0))$ and with $|x_c(0)| < \epsilon$ satisfies $x_s(t) = h(x_0(t))$ for as long as $|x_c(t)| < \epsilon$;*

(b) for every function h from (a), the equilibrium $\bar{x}_0 = 0$ of (8.1) is locally stable (asymptotically stable) [unstable] if and only if the equilibrium $\bar{x}_c = 0$ of the ODE

$$dotx_c(t) = a([x_c(t); h(x_c(t))]) \tag{8.4}$$

is locally stable (asymptotically stable) [unstable];

(c) if the equilibrium $\bar{x}_c = 0$ of (8.4) is stable then there exist constants $r > 0, \gamma > 0$ such that for every solution $x = x(t)$ of (8.1) with $|x(0)| < r$ there exists a solution $x_c = x_c(t)$ of (8.4) such that

$$\lim_{t\to\infty} e^{\gamma t}|x(t) - [x_c(t); h(x_c(t))]| = 0.$$

The set of points

$$M_c = \{\bar{x} = [\bar{x}_c; h(\bar{x}_c)] : \ |\bar{x}_c| < \epsilon\},$$

where $\epsilon > 0$ is small enough, is called the *central manifold* of (8.1). Theorem 8.3, called frequently the *center manifold theorem*, allows one to reduce the dimension of the system to be analyzed from n to q, as long as the function h defining the central manifold can be calculated exactly or to a sufficient degree of accuracy to judge local stability of (8.4).

Example 8.3 This example is taken from Sastry, p. 312. Consider system

$$\begin{aligned} \dot{x}_1(t) &= -x_1(t) + kx_2(t)^2, \\ \dot{x}_2(t) &= x_1(t)x_2(t), \end{aligned}$$

where k is a real parameter. In this case $n = 2$, $p = q = 1$, $A_c = 0$, $A_s = -1$, and k can be arbitrarily large. According to Theorem 8.3, there exists a k times differentiable function $h : \mathbf{R} \mapsto \mathbf{R}$ such that $x_1 = h(x_2)$ is an invariant manifold of the ODE (at least, in a neigborhood of the origin). Hence

$$ky^2 = h(y) + \dot{h}(y)h(y)y$$

for all sufficiently small y. For the 4th order Taylor series expansion

$$h(y) = h_2y^2 + h_3y^3 + h_4y^4 + o(y^4), \quad \dot{h}(y) = 2h_2y + 3h_3y^2 + 4h_4y^4 + o(y^3),$$

comparing the coefficients on both sides of the ODE for h yields $h_2 = k$, $h_3 = 0$, $h_4 = -2k^2$. Hence the center manifols ODE has the form

$$\dot{x}_c(t) = kx_c(t)^3 + o(x_c(t)^3),$$

which means stability for $k < 0$ and instability for $k > 0$.