Lecture 6: Storage Functions And Stability Analysis[1]

This lecture presents results describing the relation between existence of Lyapunov or storage functions and stability of dynamical systems.

6.1 Stability of an equilibria

In this section we consider ODE models

$$\dot{x}(t) = a(x(t)), \tag{6.1}$$

where $a : X \mapsto \mathbf{R}^n$ is a continuous function defined on an open subset X of $\mathbf{R}^n$. Remember that a point $\bar{x}_0 \in X$ is an *equilibrium* of (6.1) if $a(\bar{x}_0) = 0$, i.e. if $x(t) \equiv \bar{x}_0$ is a solution of (6.1). Depending on the behavior of other solutions of (6.1) (they may stay close to $\bar{x}_0$, or converge to $\bar{x}_0$ as $t \to \infty$, or satisfy some other specifications) the equilibrium may be called stable, asymptotically stable, etc. Various types of stability of equilibria can be derived using storage functions. On the other hand, in many cases existence of storage functions with certain properties is impled by stability of equilibria.

6.1.1 Locally stable equilibria

Remember that a point $\bar{x}_0 \in X$ is called a (locally) *stable equilibrium* of ODE (6.1) if for every $\epsilon > 0$ there exists $\delta > 0$ such that all maximal solutions $x = x(t)$ of (6.1) with $|x(0) - \bar{x}_0| \leq \delta$ are deinfed for all $t \geq 0$, and satisfy $|x(t) - \bar{x}_0| < \epsilon$ for all $t \geq 0$.

The statement below uses the notion of a *lower semicontinuity*: a function $f : Y \mapsto \mathbf{R}$, defined on a subset Y of $\mathbf{R}^n$, is called lower semicontinuous if

$$\lim_{r \to 0, r > 0} \inf_{\bar{x} \in Y: \ |\bar{x} - \bar{x}_*| < r} f(\bar{x}) \geq f(\bar{x}_*) \quad \forall \ \bar{x}_* \in Y.$$

---

[1]Version of September 24, 2003

Theorem 6.1 $\bar{x}_0 \in X$ *is a locally stable equilibrium of (6.1) if and only if there exist* $c > 0$ *and a lower semicontinuous function* $V : B_c(\bar{x}_0) \mapsto \mathbf{R}$, *defined on*

$$B_c(\bar{x}_0) = \{\bar{x} : \|x - \bar{x}_0\| < c\}$$

and continuous at $\bar{x}_0$, *such that* $V(x(t))$ *is monotonically non-increasing along the solutions of (6.1), and*
$$V(\bar{x}_0) < V(\bar{x}) \quad \forall \, \bar{x} \in B_c(\bar{x}_0)/\{\bar{x}_0\}.$$

Proof To prove that (ii) implies (i), define

$$\hat{V}(r) = \inf\{V(\bar{x}) - V(\bar{x}_0) : |\bar{x} - \bar{x}_0| = r$$

for $r \in (0, c)$. Since V is assumed lower semicontinuous, the infimum is actually a minimum, and hence is strictly positive for all $r \in (0, c)$. On the other hand, since V is continuous at $\bar{x}_0$, $\hat{V}(r)$ converges to zero as $r \to 0$. Hence, for a given $\epsilon > 0$, one can find $\delta > 0$ such that
$$\hat{V}(\min\{\epsilon, c/2\}) > V(\bar{x}) \quad \forall \, \bar{x} : \ |\bar{x} - \bar{x}_0| < \delta.$$

Hence a solution $x = x(t)$ of (6.1) with an initial condition such that $|x(0) - \bar{x}_0| < \delta$ (and hence $V(x(0)) < \hat{V}(\min\{\epsilon, c/2\})$ cannot cross the sphere $|\bar{x} - \bar{x}_0| = \min\{\epsilon, c/2\}$.

To prove that (i) implies (ii), define V by

$$V(\bar{x}) = \sup\{\|x(t) - \bar{x}_0\| : \ t \geq 0, \ x(0) = \bar{x}, \ x(\cdot) \text{ satisfies (6.1)} \}. \tag{6.2}$$

Since, by assumption, solutions starting close enough to $\bar{x}_0$ never leave a given disc centered at $\bar{x}_0$, V is well defined in a neigborhood X_0 of x_0. Then, by its very definition, $V(x(t))$ is not increasing for every solution of (6.1) starting in X_0. Since V is a supremum, it is lower semicontinuous (actually, here we use the fact, not mentioned before, that if $x_k = x_k(t)$ are solutions of (6.1) such that $x_k(t_0) \to \bar{x}_\infty^0$ and $x_k(t_1) \to \bar{x}_\infty^1$ then there exists a solution of (6.1) with $x(t_0) = \bar{x}_\infty^0$ and $x(t_1) = \bar{x}_\infty^1$). Moreover, V is continuous at x_0, because of stability of the equilibrium x_0. ∎

One can ask whether existence of a Lyapunov function from a better class (say, continuous functions) is possible. The answer, in general, is negative, as demonstrated by the following example.

Example 6.1 The equilibrium $\bar{x}_0 = 0$ of the first order ODE Let $a : \mathbf{R} \mapsto \mathbf{R}$ be defined by
$$a(\bar{x}) = \begin{cases} exp(-1/\bar{x}^2)\mathrm{sgn}(\bar{x}) \sin^2(\bar{x}), & \bar{x} \neq 0, \\ 0, & \bar{x} = 0. \end{cases}$$

Then a is arbitrary number of times differentialble and the equilibrium $\bar{x}_0 = 0$ of (6.1) is locally stable. However, every continuous function $V : \mathbf{R} \mapsto \mathbf{R}$ which does not increase along system trajectories will achieve a *maximum* at $\bar{x}_0 = 0$.

For the case of a *linear* system, however, local stability of equilibrium $\bar{x}_0 = 0$ implies existence of a Lyapunov function which is a positive definite quadratic form.

Theorem 6.2 *If* $a : \mathbf{R}^n \mapsto \mathbf{R}^n$ *is defined by*

$$a(\bar{x}) = A\bar{x}$$

where A *is a given n- by-n matrix, then equilibrium* $\bar{x}_0 = 0$ *of (6.1) is locally stable if and only if there exists a matrix* $Q = Q' > 0$ *such that* $V(x(t)) = x(t)'Qx(t)$ *is monotonically non-increasing along the solutions of (6.1).*

The proof of this theorem, which can be based on considering a Jordan form of A, is usually a part of a standard linear systems class.

6.1.2 Locally asymptotically stable equilibria

A point $\bar{x}_0$ is called a (locally) *asymptotically stable equilibrium* of (6.1) if it is a stable equilibria, and, in addition, there exists $e_0 > 0$ such that every solution of (6.1) with $|x(0) - \bar{x}_0| < \epsilon_0$ converges to $\bar{x}_0$ as $t \to \infty$.

Theorem 6.3 *If* $V : X \mapsto \mathbf{R}$ *is a* continuous function *such that*

$$V(\bar{x}_0) < V(\bar{x}) \quad \forall \, \bar{x} \in X/\{\bar{x}_0\},$$

and $V(x(t))$ *is strictly monotonically decreasing for every solution of (6.1) except* $x(t) \equiv \bar{x}_0$ *then* $\bar{x}_0$ *is a locally asymptotically stable equilibrium of (6.1).*

Proof From Theorem 6.1, $\bar{x}_0$ is a locally stable equilibrium. It is sufficient to show that every solution $x = x(t)$ of (6.1) starting sufficiently close to $\bar{x}_0$ will converge to $\bar{x}_0$ as $t \to \infty$. Assume the contrary. Then $x(t)$ is bounded, and hence will have at least one limit point $\bar{x}_*$ which is not $\bar{x}_0$. In addition, the limit $\bar{V}$ of $V(x(t))$ will exist. Consider a solution $x_* = x_*(t)$ starting from that point. By continuous dependence on initial conditions we conclude that $V(x_*(t)) = \bar{V}$ is constant along this solution, which contradicts the assumptions. ∎

A similar theorem deriving existence of a smooth Lyapunov function is also valid.

Theorem 6.4 *If* $\bar{x}_0$ *is an asymptotically stable equilibrium of system (6.1) where* $a : X \mapsto \mathbf{R}^n$ *is a continuously differentiable function defined on an open subset* X *of* $\mathbf{R}^n$ *then there exists a continuously differentiable function* $V : B_\epsilon(\bar{x}_0) \mapsto \mathbf{R}$ *such that* $V(\bar{x}_0) < V(\bar{x})$ *for all* $\bar{x} \neq \bar{x}_0$ *and*

$$\nabla V(\bar{x})a(\bar{x}) < 0 \quad \forall \, \bar{x} \in B_\epsilon(\bar{x}_0)/\{\bar{x}_0\}.$$

Proof Define V by

$$V(x(0)) = \int_0^\infty \rho(|x(t)|^2)dt,$$

where $\rho : [0, \infty) \mapsto [0, \infty)$ is positive for positive arguments and continuously differentiable. If V is correctly defined and differentiable, differentiation of $V(x(t))$ with respect to t at $t = 0$ yields

$$\nabla V(x(0))a(x(0)) = -\rho(|x(0)|^2),$$

which proves the theorem. To make the integral convergent and continuously differentiable, it is sufficient to make $\rho(y)$ converging to zero quickly enough as $y \to 0$. ∎

For the case of a *linear* system, a classical Lyapunov theorem shows that local stability of equilibrium $\bar{x}_0 = 0$ implies existence of a strict Lyapunov function which is a positive definite quadratic form.

Theorem 6.5 *If* $a : \mathbf{R}^n \mapsto \mathbf{R}^n$ *is defined by*

$$a(\bar{x}) = A\bar{x}$$

where A is a given n- by-n matrix, then equilibrium $\bar{x}_0 = 0$ of (6.1) is locally asymptotically stable if and only if there exists a matrix $Q = Q' > 0$ such that, for $V(\bar{x}) = \bar{x}'Q\bar{x}$,

$$\nabla V(\bar{x})A\bar{x} = -|\bar{x}|^2.$$

6.1.3 Globally asymptotically stable equilibria

Here we consider the case when $a : \mathbf{R}^n \mapsto \mathbf{R}^n$ in defined for all vectors. An equilibrium $\bar{x}_0$ of (6.1) is called *globally asymptotically stable* if it is locally stable and *every* solution of (6.1) converges to $\bar{x}_0$ as $t \to \infty$.

Theorem 6.6 *If function $V : \mathbf{R}^n \mapsto \mathbf{R}$ has a unique minimum at $\bar{x}_0$, is strictly monotonically decreasing along every trajectory of (6.1)except $x(t) \equiv \bar{x}_0$, and has* bounded *level sets then $\bar{x}_0$ is a globally asymptotically stable equilibrium of (6.1).*

The proof of the theorem follows the lines of the proof of Theorem 6.4. Note that the assumption that the level sets of V are bounded is critically important: without it, some solutions of (6.1) may converge to infinity instead of $\bar{x}_0$.