# 6.241 Spring 2011

Final Exam

5/16/2011, 9:00am — 12:00pm

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The test is open books/notes, but no collaboration is allowed: i.e., you should not discuss this exam or solution approaches with anybody, except for the teaching staff.

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## Problem 1

Let $(A, b, c, 0)$ be a state-space model of a LTI system, with $A \in \mathbb{R}^{n\times n}$, $b, c' \in \mathbb{R}^n$. Assume that $\lambda_i(A) + \lambda_j(A) \neq 0$, for all $i$, $j$. Consider the equation

$$AX + XA + bc = 0;$$

show that there exists a non-singular matrix $X$ that satisfies the equation if and only if $(A, b)$ is controllable and $(c, A)$ is observable.

## Problem 2

Consider a LTI system described by the following state-space model:

$$A = \begin{bmatrix} -1 & 2 & 2 \\ -3 & -1 & -3 \\ 3 & -2 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & -1 & 0 \end{bmatrix}, \quad D = 0.$$

1. Construct a Kalman decomposition for this system, and compute the transfer function of the system. Is the system controllable/stabilizable, observable/detectable?
2. Design a stabilizing model-based compensator (i.e., composed of a full-state controller and an observer).
3. What is the transfer function of the compensator? Can you give a "classical" interpretation of the control law?

## Problem 3

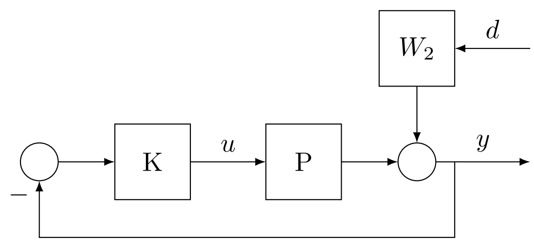
Consider a plant with transfer function $G(s) = 1/(s-1)$. Find all feedback compensators $K(s)$ such that (i) the closed-loop system is stable, and (ii) the output response to a unit step disturbance at the output is asymptotically zero.
*(Recall that, assuming the closed-loop transfer function $T_{yd}$ is stable, then* $\lim_{t\to+\infty} y(t) = \lim_{s\to 0} sT_{yd}(s)D(s)$*, where $D(s)$ is the Laplace transform of the input signal.)*

## Problem 4

Consider the block diagram shown below. $P$ is an uncertain SISO plant with transfer function $P(s) = P_0(s) + W_1(s)\Delta_1(s)$, where $W_1(s)$ is a stable transfer function, $P(s)$ and $P_0(s)$ have the same number of right half-plane poles, and

$$|\mathrm{Re}[\Delta_1(s)]| \le \alpha, \qquad |\mathrm{Im}[\Delta_1(s)]| \le \beta, \qquad \forall s \in \mathbb{C}.$$

The transfer function $W_2(s)$ is a stable frequency weight.




1. Derive necessary and sufficient conditions for robust stability, i.e., such that the closed-loop system shown in the figure is externally stable for all admissible $\Delta_1(s)$.

2. Assume $P = P_0$. Derive necessary and sufficient conditions for nominal performance, i.e., to ensure that $\|y\| \le \|d\|$, for all square-integrable disturbance inputs $d \in \mathcal{L}_2$.

3. Derive necessary and sufficient conditions for robust stability and performance, i.e., such that the closed loop system is stable, and $\|y\| \le \|d\|$, for all square-integrable disturbance inputs $d \in \mathcal{L}_2$, and for any admissible $\Delta_1(s)$.

4. Can you give a graphical interpretation of these conditions?

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