6.045: Automata, Computability, and Complexity
Or, Great Ideas in Theoretical Computer Science
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Class 7

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Today

- Basic computability theory
- Topics:
  - Decidable and recognizable languages
  - Recursively enumerable languages
  - Turing Machines that solve problems involving FAs
  - Undecidability of the Turing machine acceptance problem
  - Undecidability of the Turing machine halting problem
- Reading: Sipser, Sections 3.1, 3.2, Chapter 4
- Next: Sections 5.1, 5.2

Decidable and Recognizable Languages

Decidable and recognizable languages

- Last time, we began studying the important notion of computability.
- As a concrete model of computation, we introduced basic one-tape, one-head Turing machines.
- Also discussed some variants.
- Claimed they are all equivalent, so the notion of computability is robust.
- Today: Look more carefully at the notions of computability and equivalence.

Decidable and recognizable languages

- Assume: TM has accepting state q_{acc} and rejecting state q_{rej}.
- Definition: TM M recognizes language L provided that L = { w | M on w reaches q_{acc} } = { w | M accepts w }.
- Another important notion of computability:
- Definition: TM M decides language L provided that both of the following hold:
  - On every w, M eventually reaches either q_{acc} or q_{rej}.
  - L = { w | M on w reaches q_{acc} }.
- Thus, if M recognizes L, then:
  - Words in L lead to q_{acc}.
  - Words not in L either lead to q_{rej} or never halt ("loop").
- Whereas if M decides L, then:
  - Words in L lead to q_{acc}.
  - Words not in L lead to q_{rej}.

Always halts

Decidable and recognizable languages

- Theorem 1: If M decides L then M recognizes L.
- Obviously.
- But not necessarily vice versa.
- In fact, these two notions define different language classes:
- Definition:
  - L is Turing-recognizable if there is some TM that recognizes L.
  - L is Turing-decidable if there is some TM that decides L.
- The classes of Turing-recognizable and Turing-decidable languages are different.
- Theorem 2: If L is Turing-decidable then L is Turing-recognizable.
- Obviously.
- But the other direction does not hold---there are languages that are Turing-recognizable but not Turing-decidable.
- We’ll see some examples soon.

Decidable and recognizable languages

- Theorem 3: If L is Turing-decidable then L^c is T-decidable.
- Proof:
  - Suppose that M decides L.
  - Design a new machine M′ that behaves just like M, but:
    - If M accepts, M′ rejects.
    - If M rejects, M′ accepts.
  - Formally, can do this by interchanging q_{acc} and q_{rej}.
  - Then M′ decides L^c.

Decidable and recognizable languages

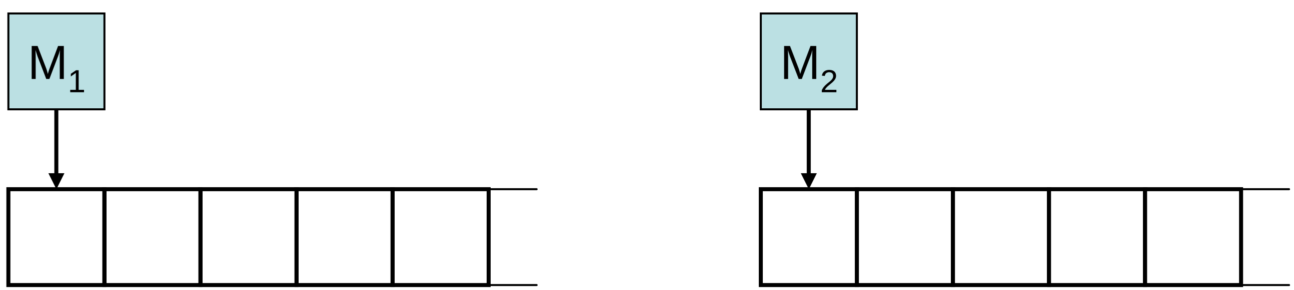
- A basic connection between Turing-recognizable and Turing-decidable languages:
- Theorem 4: L is Turing decidable if and only if L and L^c are both Turing-recognizable.
- Proof: $\Rightarrow$
  - Suppose that L is Turing-decidable.
  - Then L is Turing-recognizable, by Theorem 2.
  - Also, L^c is Turing-decidable, by Theorem 3.
  - So L^c is Turing-recognizable, by Theorem 2.
- Proof: $\Leftarrow$
  - Given M_1 recognizing L, and M_2 recognizing L^c.
  - Produce a Turing Machine M that decides whether or not its input w is in L or L^c.

Decidable and recognizable languages

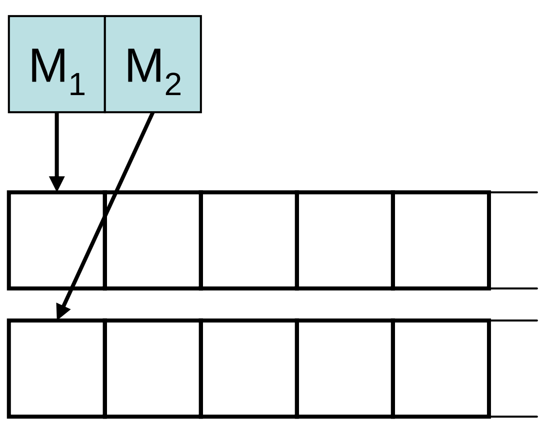
- Theorem 4: L is Turing decidable if and only if L and L^c are both Turing-recognizable.
- Proof: $\Leftarrow$
  - Given M_1 recognizing L, and M_2 recognizing L^c.
  - Produce a Turing Machine M that decides whether or not its input w is in L or L^c.
  - Idea: Run both M_1 and M_2 on w.
    - One must accept.
    - If M_1 accepts, then M accepts.
    - If M_2 accepts, then M rejects.
  - But, we can't run M_1 and M_2 one after the other because the first one might never halt.
  - Run them in parallel, until one accepts?
  - How? We don't have a parallel Turing Machine model.

Decidable and recognizable languages

- Theorem 4: L is Turing decidable if and only if L and L^c are both Turing-recognizable.
- Proof: $\Leftarrow$
  - M_1 recognizes L, and M_2 recognizes L^c.

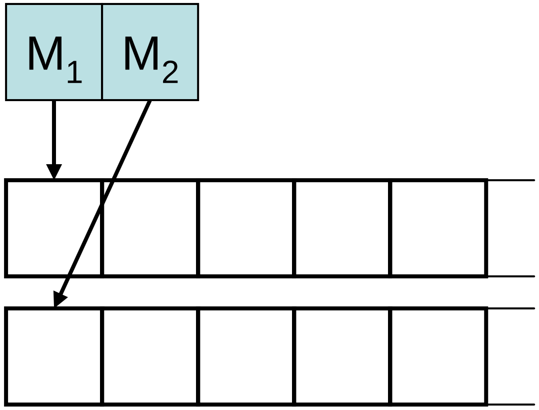


  - Let M be a 2-tape Turing Machine:



Decidable and recognizable languages

- Theorem 4: L is Turing decidable if and only if L and L^c are both Turing-recognizable.
- Proof: $\Leftarrow$
  - M copies input from 1st tape to 2nd tape.
  - Then emulates M_1 and M_2 together, step-by-step.
  - No interaction between them.
  - M's finite-state control keeps track of states of M_1 and M_2; thus, Q includes $Q_1 \times Q_2$.
  - Also includes new start, accept, and reject states and whatever else is needed for bookkeeping.



Language Classification

- Four possibilities:
  - L and L^c are both Turing-recognizable.
    - Equivalently, L is Turing-decidable.
  - L is Turing-recognizable, L^c is not.
  - L^c is Turing-recognizable, L is not.
  - Neither L nor L^c is Turing-recognizable.
- All four possibilities occur, as we will see.
- How do we know that there are languages L that are neither Turing-recognizable nor co-Turing-recognizable?
- Cardinality argument:
  - There are uncountably many languages.
  - There are only countably many Turing-recognizable languages and only countably many co-Turing-recognizable languages.
  - Because there are only countably many Turing machines (up to renaming).

Examples

- Example: Every regular language L is decidable.
  - Let M be a DFA with L(M) = L.
  - Design a Turing machine M′ that simulates M.
  - If, after processing the input, the simulated M is in an accepting state, M′ accepts; else M′ rejects.

Examples

- Example: Let X = be the set of binary representations of natural numbers for which the following procedure halts:

  while x $\neq$ 1 do

  if x is odd then x := 3x + 1

  if x is even then x := x/2

  halt

  – Obviously, X is Turing-recognizable: just simulate this procedure and accept if/when it halts.

  – Is it decidable? (?)

Closure Properties

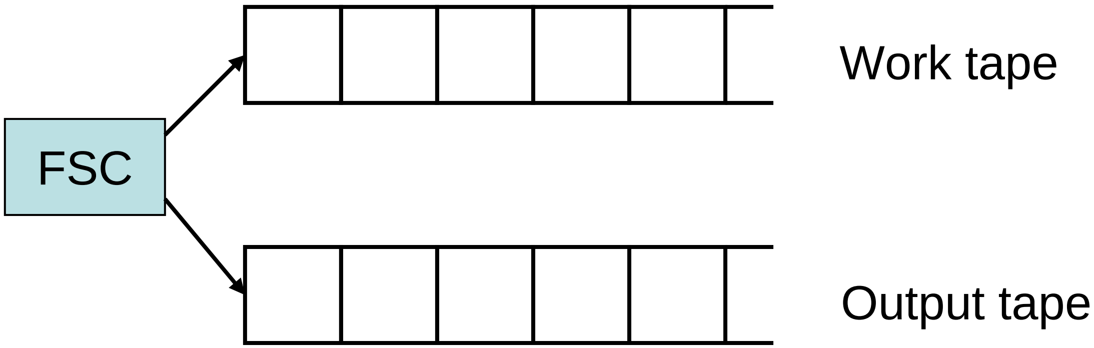
- Theorem 5: The set of Turing-recognizable languages is closed under set union and intersection.
- Proof:
  - Run both machines in parallel.
  - For union, accept if either accepts.
  - For intersection, accept if both accept.

- However, the set of Turing-recognizable languages is not closed under complement.
- As we will soon see.
- Theorem 6: The set of Turing-decidable languages is closed under union, intersection, and complement.
- Theorem 7: Both the Turing-recognizable and Turing-decidable languages are closed under concatenation and star (HW).

Recursively Enumerable Languages

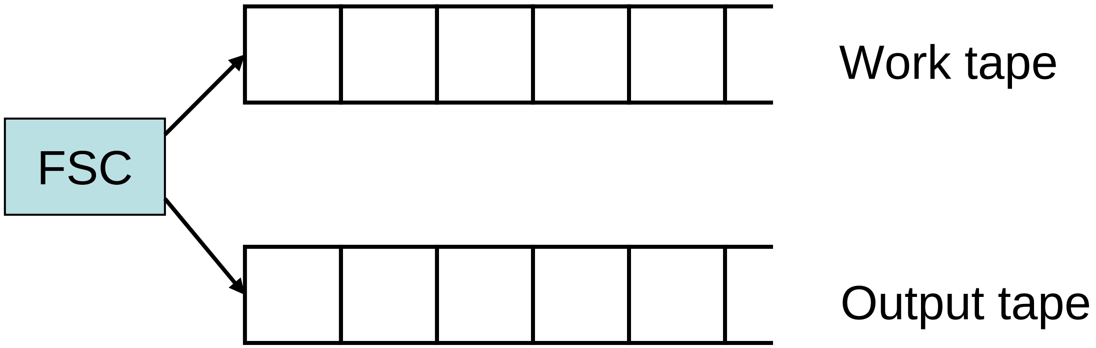
Recursively enumerable languages

- Yet another kind of computability for Turing Machines.
- An enumerator is a Turing Machine variant:



- Starts with a blank work tape (no input).
- Prints a sequence of finite strings (possibly infinitely many) on output tape.
- More specifically, e.g.:
  - Enters a special state q_{print}, where contents of work tape, up to first blank, are copied to output tape, followed by blank as a separator.
  - Then machine continues.
  - No accept or reject states.

Recursively enumerable languages



- Starts with a blank work tape (no input).
- Prints a sequence of finite strings (possibly infinitely many) on output tape.
- It may print the same string more than once.
- If E is an enumerator, then define

$$L(E) = \{ x \mid x \text{ is printed by } E \}.$$

- If $L = L(E)$ for some enumerator E, then we say that L is recursively enumerable (r.e.).

Recursively enumerable languages

- Interesting connection between recursive enumerability and Turing recognizability:
- Theorem 8: L is recursively enumerable if and only if L is Turing-recognizable.
- Proof: $\Rightarrow$
  - Given E, an enumerator for L, construct Turing machine M to recognize L.
  - M: On input x:
    - M simulates E (on no input, as usual).
    - Whenever E prints, M checks to see if the new output is x.
    - If it ever sees x, M accepts.
    - Otherwise, M keeps going forever.

Recursively enumerable languages

- Theorem 8: L is recursively enumerable if and only if L is Turing-recognizable.
- Proof: $\Leftarrow$
  - Given M, a Turing machine that recognizes L, construct E to enumerate L.
  - Idea:
    - Simulate M on all inputs.
    - If/when any simulated execution reaches q_{acc}, print out the associated input.
  - As before, we can't run M on all inputs sequentially, because some computations might not terminate.
  - So we must run them in parallel.
  - But this time we must run infinitely many computations, so we can't just use a multitape Turing machine.

Recursively enumerable languages

- Theorem 8: L is recursively enumerable if and only if L is Turing-recognizable.
- Proof: $\Leftarrow$
  - Given M, a Turing machine that recognizes L, construct E to enumerate L.
  - Simulate M on all inputs; when any simulated execution reaches q_{acc}, print out the associated input.
  - New trick: Dovetailing
    - Run 1 step for 1st input string, ε.
    - Run 2 steps for 1st and 2nd inputs, ε and 0.
    - Run 3 steps for 1st, 2nd, and 3rd inputs, ε, 0 and 1.
    - …
    - Run more and more steps for more and more inputs.
  - Eventually succeeds in reaching q_{acc} for each accepting computation of M, so enumerates all elements of L.

Recursively enumerable languages

- Theorem 8: L is recursively enumerable if and only if L is Turing-recognizable.
- Proof: $\Leftarrow$
  - Simulate M on all inputs; when any simulated execution reaches q_{acc}, print out the associated input.
  - Dovetail all computations of M.
  - Complicated bookkeeping, messy to work out in detail.
  - But can do algorithmically, hence on a Turing machine.

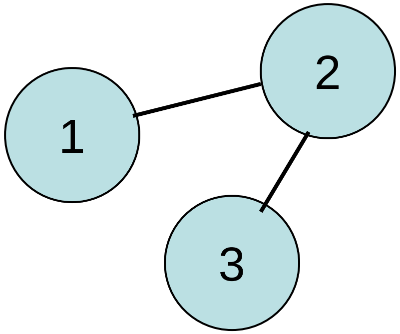
Turing Machines that solve problems for other domains besides strings

Turing Machines that solve problems for other domains

- [Sipser Section 4.1]
- Our examples of computability by Turing machines have so far involved properties of strings, and numbers represented by strings.
- We can also consider computability by TMs for other domains, such as graphs or DFAs.
- Graphs:
  - Consider the problem of whether a given graph has a cycle of length > 2.
  - Can formalize this problem as a language (set of strings) by encoding graphs as strings over some finite alphabet.
  - Graph = (V,E), V = vertices, E = edges, undirected.

Turing Machines that solve graph problems

- Consider the problem of whether a given graph has a cycle of length > 2.
- Formalize as a language (set of strings) by encoding graphs as strings over some finite alphabet.
- Graph = (V,E), V = vertices, E = edges, undirected.
- A standard encoding:
  - Vertices = positive integers (represented in binary)
  - Edges = pairs of positive integers
  - Graph = list of vertices, list of edges.
- Example: ((1, 2, 3), ((1, 2), (2, 3)))
- Write <G> for the encoding of G.

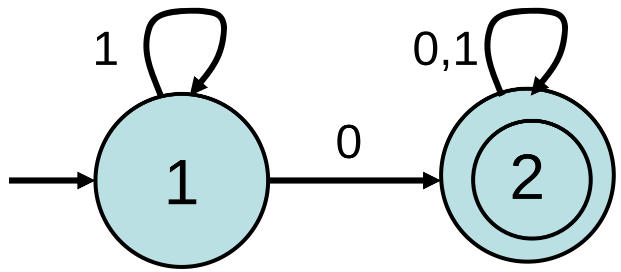


Turing Machines that solve graph problems

- Consider the problem of whether a given graph has a cycle of length > 2.
- Graph = (V,E), V = vertices, E = edges, undirected.
- Write <G> for the encoding of G.
- Using this representation for the input, we can write an algorithm to determine whether or not a given graph G has a cycle, and formalize the algorithm using a Turing machine.
  - E.g., search and look for repeated vertices.
- So cyclicity is a decidable property of graphs.

Turing Machines that solve problems for other domains

- We can also consider computability for domains that are sets of machines:
- DFAs:
  - Encode DFAs using bit strings, by defining standard naming schemes for states and alphabet symbols.
  - Then a DFA tuple is again a list.
  - Example:



Encode as:

$$(\underbrace{(1, 2)}_{Q}, \underbrace{(0, 1)}_{\Sigma}, \underbrace{((1, 1, 1), (1, 0, 2), (2, 0, 2), (2, 1, 2))}_{\delta}, \underbrace{(1)}_{q_0}, \underbrace{(2)}_{F})$$

  - Encode the list using bit strings.
  - Write <M> for the encoding of M.
  - So we can define languages whose elements are (bit strings representing) DFAs.

Turing Machines that solve DFA problems

- Example: $L_1 = \{ < M > \mid L(M) = \emptyset \}$ is Turing-decidable
- Elements of L_1 are bit-string representations of DFAs that accept nothing (emptiness problem).
- Already described an algorithm to decide this, based on searching to determine whether any accepting state is reachable from the start state.
- Could formalize this (painfully) as a Turing machine.
- Proves that L_1 is Turing-decidable.

- Similarly, all the other decision problems we considered for DFAs, NFAs, and regular expressions are Turing-decidable (not just Turing-recognizable).
- Just represent the inputs using standard encodings and formalize the algorithms that we've already discussed, using Turing machines.

Turing Machines that solve DFA problems

- Example: Equivalence for DFAs

  $L_2 = \{ < M_1, M_2 > \mid L(M_1) = L(M_2) \}$ is Turing-decidable.
- Elements of L_2 are bit-string representations of pairs of DFAs that recognize the same language.
- Note that the domain we encode is pairs of DFAs.
- Already described an algorithm to decide this, based on testing inclusion both ways; to test whether $L(M_1) \subseteq L(M_2)$, just test whether $L(M_1) \cap (L(M_2))^c = \emptyset$.
- Formalize as a Turing machine.
- Proves that L_2 is Turing-decidable.

Turing Machines that solve DFA problems

- Example: Acceptance for DFAs

  $L_3 = \{ < M, w > \mid w \in L(M) \}$ is Turing-decidable.
- Domain is (DFA, input) pairs.
- Algorithm simply runs M on w.
- Formalize as a Turing machine.
- Proves that L_3 is Turing-decidable.

Moving on…

- Now, things get more complicated: we consider inputs that are encodings of Turing machines rather than DFAs.
- In other words, we will discuss Turing machines that decide questions about Turing machines!

Undecidability of the Turing Machine Acceptance Problem

Undecidability of TM Acceptance Problem

- Now (and for a while), we will focus on showing that certain languages are not Turing-decidable, and that some are not even Turing-recognizable.
- It's easy to see that such languages exist, based on cardinality considerations.
- Now we will show some specific languages are not Turing decidable, and not Turing-recognizable.
- These languages will express questions about Turing machines.

Undecidability of TM Acceptance

- We have been discussing decidability of problems involving DFAs, e.g.:
  - { < M > | M is a DFA and L(M) = ∅ }, decidable by Turing machine that searches M's digraph.
  - { < M, w > | M is a DFA, w is a word in M's alphabet, and w ∈ L(M) }, decidable by a Turing machine that emulates M on w.
- Turing machines compute only on strings, but we can regard them as computing on DFAs by encoding the DFAs as strings (using a standard encoding).
- Now we consider encoding Turing machines as strings, and allowing other Turing machines to compute on these strings.
- Encoding of Turing machines: Standard state names, lists, etc., similar to DFA encoding.
- <M> = encoding of Turing machine M.
- <M, w> = encoding Turing machine + input string
- Etc.

Problems we will consider

- Acc_{TM} = { < M, w > | M is a (basic) Turing machine, w is a word in M's alphabet, and M accepts w }.
- Halt_{TM} = { < M, w > | M is a Turing machine, w is a word in M's alphabet, and M halts (either accepts or rejects) on w }.
- Empty_{TM} = { < M > | M is a Turing machine and L(M) = $\emptyset$ }
  - Recall: L(M) refers to the set of strings M accepts.
- Etc.

- Thus, we can formulate questions about Turing machines as languages.
- Then we can ask if they are Turing-decidable; that is, can some particular TM answer these questions about all (basic) TMs?
- We'll prove that they cannot.

The Acceptance Problem

- Acc_{TM} = { < M, w > | M is a (basic) Turing machine and M accepts w }.
- Theorem 1: Acc_{TM} is Turing-recognizable.
- Proof:
  - Construct a TM U that recognizes Acc_{TM}.
  - U: On input < M, w >:
    - Simulate M on input w.
    - If M accepts, accept.
    - If M rejects, reject.
    - Otherwise, U loops forever.
  - Then U accepts exactly < M, w> encodings for which M accepts w.

- U is sometimes called a universal Turing machine because it runs all TMs.
  - Like an interpreter for a programming language.

The Acceptance Problem

- Acc_{TM} = { < M, w > | M is a TM and M accepts w }.
- U: On input < M, w >:
  - Simulate M on input w.
  - If M accepts, accept.
  - If M rejects, reject.
  - Otherwise, U loops forever.
- U recognizes Acc_{TM}.
- U is a universal Turing machine because it runs all TMs.
- U uses a fixed, finite set of states, and set of alphabet symbols, but still simulates TMs with arbitrarily many states and symbols.
  - All encoded using the fixed symbols, decoded during emulation.

The Acceptance Problem

- Acc_{TM} = { < M, w > | M is a TM and M accepts w }.
- U: On input < M, w >:
  - Simulate M on input w.
  - If M accepts, accept.
  - If M rejects, reject.
  - Otherwise, U loops forever.
- U recognizes Acc_{TM}.
- Does U decide Acc_{TM} ?
- No.
  - If M loops forever on w, U loops forever on <M,w>, never accepts or rejects.
  - To decide, U would have to detect when M is looping and reject.
  - Seems difficult…

Undecidability of Acceptance

- Theorem 2: Acc_{TM} is not Turing-decidable.
- Proof:
  - Assume that Acc_{TM} is Turing-decidable and produce a contradiction.
  - Similar to the diagonalization argument that shows that we can't enumerate all languages.
  - Since (we assume) Acc_{TM} is Turing-decidable, there must be a particular TM H that decides Acc_{TM}:
    - H(<M,w>):
      - accepts if M accepts w,
      - rejects if M rejects w,
      - rejects if M loops on w.

Undecidability of Acceptance

- Theorem 2: Acc_{TM} is not Turing-decidable.
- Proof, cont'd:
  - H(<M,w>) accepts if M accepts w, rejects if M rejects w or if M loops on w.
  - Use H to construct another TM H′ that decides a special case of the same language.
  - Instead of considering whether M halts on an arbitrary w, just consider M on its own representation:
  - H′(<M>):
    - accepts if M accepts <M>,
    - rejects if M rejects <M> or if M loops on <M>.
  - If H exists, then so does H′: H′ simply runs H on certain arguments.

Undecidability of Acceptance

- Theorem 2: Acc_{TM} is not Turing-decidable.
- Proof, cont'd:
  - H′(<M>):
    - accepts if M accepts <M>,
    - rejects if M rejects <M> or if M loops on <M>.
  - Now define D (the diagonal machine) to do the opposite of H′:
  - D(<M>):
    - rejects if M accepts <M>,
    - accepts if M rejects <M> or if M loops on <M>.
  - If H′ exists, then so does D: D runs H′ and outputs the opposite.

Undecidability of Acceptance

- Theorem 2: Acc_{TM} is not Turing-decidable.
- Proof, cont'd:
  - D(<M>):
    - rejects if M accepts <M>,
    - accepts if M rejects <M> or if M loops on <M>.
  - Now, what happens if we run D on <D>?
  - Plug in D for M:
  - D(<D>):
    - rejects if D accepts <D>,
    - accepts if D rejects <D> or if D loops on <D>.
  - Then D accepts <D> if and only if D does not accept <D>, contradiction!
  - So Acc_{TM} is not Turing-decidable.
  - !!!

Diagonalization Proofs

- This undecidability proof for Acc_{TM} is an example of a diagonalization proof.
- Earlier, we used diagonalization to show that the set of all languages is not countable.
- Consider a big matrix, with TMs labeling rows and strings that represent TMs labeling columns.
- The major diagonal describes results for M(<M>), for all M.
- D is a diagonal machine, constructed explicitly to differ from the diagonal entries: D(<M>)'s result differs from M(<M>)'s.
- Implies that D itself can't appear as a label for a row in the matrix, a contradiction since the matrix is supposed to include all TMs.

Summary: Acc_{TM}

- We have shown that Acc_{TM} = { < M, w > | M is a Turing machine and M accepts w } is Turing-recognizable but not Turing-decidable.
- Corollary: $(Acc_{TM})^c$ is not Turing-recognizable.
- Proof:
  - By Theorem 4.
  - If Acc_{TM} and $(Acc_{TM})^c$ were both Turing-recognizable, then Acc_{TM} would be Turing-decidable.

Undecidability of the Turing Machine Halting Problem

The Halting Problem

- Halt_{TM} = { < M, w > | M is a Turing machine and M halts on (either accepts or rejects) w }.
- Compare with Acc_{TM} = { < M, w > | M is a Turing machine and M accepts w }.
- Terminology caution: Sipser calls Acc_{TM} the "halting problem", and calls Halt_{TM} just Halt_{TM}.
- Theorem: Halt_{TM} is not Turing-decidable.
- Proof:
  - Let's not use diagonalization.
  - Rather, take advantage of diagonalization work already done for Acc_{TM}, using new method: reduction.
  - Prove that, if we could decide Halt_{TM}, then we could decide Acc_{TM}.
  - Reduction is a very powerful, useful technique for showing undecidability; we'll use it several times.
  - Also useful (later) to show inherent complexity results.

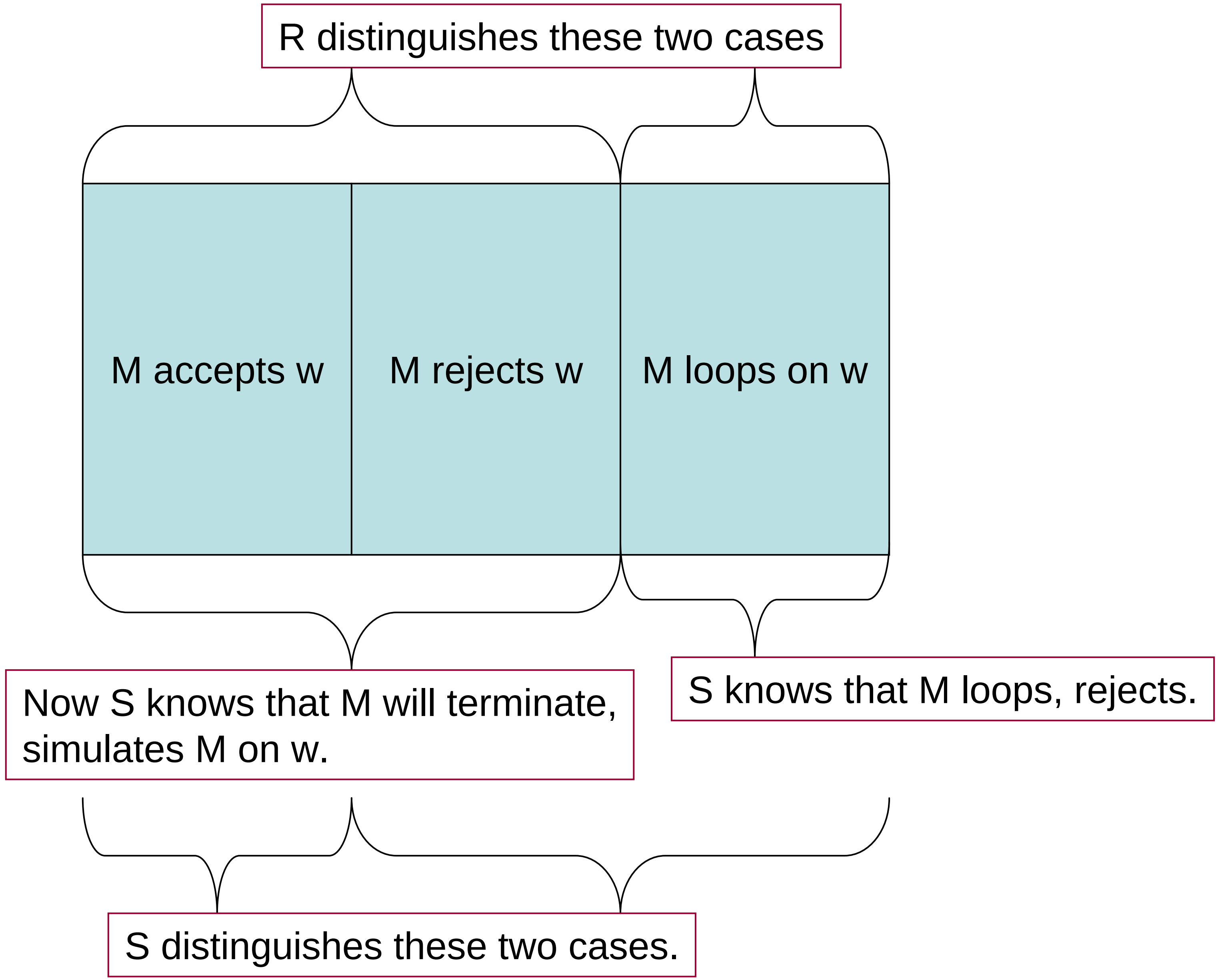
The Halting Problem

- Halt_{TM} = { < M, w > | M halts on (accepts or rejects) w }.
- Theorem: Halt_{TM} is not Turing-decidable.
- Proof:
  - Suppose for contradiction that Halt_{TM} is Turing-decidable, say by Turing machine R:
    - R(<M,w>):
      - accepts if M halts on (accepts or rejects) w,
      - rejects if M loops (neither accepts nor rejects) on w.
  - Using R, define new TM S to decide Acc_{TM}:
    - S: On input <M,w>:
      - Run R on <M,w>; R must either accept or reject; can't loop, by definition of R.
      - If R accepts then M must halt (accept or reject) on w. Then simulate M on w, knowing this must terminate. If M accepts, accept. If M rejects, reject.
      - If R rejects, then reject.

The Halting Problem

- Theorem: $Halt_{TM}$ is not Turing-decidable.
- Proof:
  - Suppose $Halt_{TM}$ is Turing-decidable by TM R.
    - S: On input <M,w>:
      - Run R on <M,w>; R must either accept or reject; can't loop, by definition of R.
      - If R accepts then M must halt (accept or reject) on w. Then simulate M on w, knowing this must terminate. If M accepts, accept. If rejects, reject.
      - If R rejects, then reject.
  - Claim S decides Acc_{TM}: 3 cases:
    - If M accepts w, then R accepts <M,w>, and the simulation leads S to accept.
    - If M rejects w, then R accepts <M,w>, and the simulation leads S to reject.
    - If M loops on w, then R rejects <M,w>, and S rejects.
    - That's what's supposed to happen in three cases, for Acc_{TM}.

The Three Cases



The Halting Problem

- Theorem: $Halt_{TM}$ is not Turing-decidable.
- Proof:
  - Suppose $Halt_{TM}$ is Turing-decidable by TM R.
  - S: On input <M,w>:
    - Run R on <M,w>; R must either accept or reject; can't loop, by definition of R.
    - If R accepts then M must halt (accept or reject) on w. Then simulate M on w, knowing this must terminate. If M accepts, accept. If rejects, reject.
    - If R rejects, then reject.
  - S decides Acc_{TM}.
  - So Acc_{TM} is decidable, contradiction.
  - Therefore, $Halt_{TM}$ is not Turing-decidable.

The Halting Problem

- Theorem: $Halt_{TM}$ is not Turing-decidable.
- Also:
- Theorem: $Halt_{TM}$ is Turing-recognizable.
- So:
- Corollary: $(Halt_{TM})c$ is not Turing-recognizable.

Next time…

- More undecidable problems:
  - About Turing machines:
    - Emptiness, etc.
  - About other things:
    - Post Correspondence Problem (a string matching problem).
- Reading: Sipser Sections 4.2, 5.1.