6.041SC Probabilistic Systems Analysis and Applied Probability, Fall 2013 Transcript – Recitation: Convergence in Probability Example

In this problem, we're given a random variable X which has a uniform distribution in the interval negative 1 to 1. In other words, if we were to draw out the PDF of X, we see that in the interval negative 1 to 1, it has value 1/2. Now we're given a sequence random variables X1, X2, and so on, where each Xi has the same distribution as X and different Xi's are independent.

For part a, we would like to know if the sequence Xi converges to some number-- let's call it c-- in probability as i goes to infinity-- whether this is true. Let's first recall the definition of convergence in probability. If this does happen, then by definition, we'll have that for every epsilon greater than 0, the probability Xi minus c greater equal to epsilon, this quantity will go to 0 in the limit of i going to infinity. In other words, with very high probability, we will find Xi to be very concentrated around the number c if this were to be the PDF of Xi.

Now, can this be true? Well, we know that each Xi is simply a uniform distribution over negative 1 to 1. It doesn't really change as we increase i. So intuitively, the concentration around any number c is not going to happen. So we should not expect a convergence in probability in this sense.

For part b, we would like to know whether the sequence Yi, defined as Xi divided by i, converges to anything in probability. Well, by just looking at the shape of Yi, we know that since the absolute value of Xi is less than 1, then we expect the absolute value of Yi is less than 1/i. So eventually, Yi gets very close to 0 as i goes to infinity. So it's safe to bet that maybe Yi will converge to 0 in probability.

Let's see if this is indeed the case. The probability of Yi minus 0 greater equal to epsilon is equal to the probability of Yi absolute value greater equal to epsilon. Now, previously we know that the absolute value of Yi is at most 1/i by the definition of Yi. And hence the probability right here is upper bounded by the probability of 1i greater equal to epsilon.

Notice in this expression, there is nothing random. i is simply a number. Hence this is either 1 if i is less equal to 1/epsilon, or 0 if i is greater than 1/epsilon.

Now, this tells us, as long as i is great enough-- it's big enough compared to epsilon-- we know that this quantity here is [INAUDIBLE] 0. And that tells us in the limit of i goes to infinity probability of Yi deviating from 0 by more than epsilon goes to 0. And that shows that indeed, Yi converges to 0 in probability because the expression right here, this limit, holds for all epsilon.

Now, in the last part of the problem, we are looking at a sequence Zi defined by Xi raised to the i-th power. Again, since we know Xi is some number between negative 1 and 1, this number raised to the i-th power is likely to be very small. And likely to be small in the sense that it will have absolute value close to 0. So a safe guess will be the sequence Zi converges to 0 as well as i goes to infinity.

How do we prove this formally? We'll start again with a probability that Zi stays away from 0 by more than epsilon and see how that evolves. And this is equal to the probability that Xi raised to the i-th power greater equal to epsilon. Or again, we can write this by taking out the absolute value that Xi is less equal to negative epsilon raised to the 1 over i-th power or Xi greater equal to epsilon 1 over i-th power.

So here, we'll divide into two cases, depending on the value of epsilon. In the first case, epsilon is greater than 1. Well, if that's the case, then we know epsilon raised to some positive power is still greater than 1. But again, Xi cannot have any positive density be on the interval negative 1 or 1. And hence we know the probability above, which is Xi less than some number smaller than negative 1 or greater than some number bigger than 1 is 0. So that case is handled.

Now let's look at a case where epsilon is less than 1, greater than 0. So in this case, epsilon to the 1/i will be less than 1. And it's not that difficult to check that since Xi has uniform density between negative 1 and 1 of magnitude 1/2, then the probability here was simply 2 times 1/2 times the distance between epsilon to the 1 over i-th power and 1.

So in order to prove this quantity converge to 0, we simply have to justify why does epsilon to the 1/i converge to 1 as i goes to infinity. For that, we'll recall the properties of exponential functions. In particular, if a is a positive number and x is its exponent, if we were to take the limit as x goes to 0 and look at the value of a to the power of x, we see that this goes to 1.

So in this case, we'll let a be equal to epsilon and x be equal to 1/i. As we can see that as i goes to infinity, the value of x, which is 1/i, does go to 0. And therefore, in the limit i going to infinity, the value of epsilon to the 1 over i-th power goes to 1.

And that shows if we plug this limit into the expression right here that indeed, the term right here goes to 0 as i goes to infinity. And all in all, this implies the probability of Zi minus 0 absolute value greater equal to epsilon in the limit of i going to infinity converges to 0 for all positive epsilon. And that completes our proof that indeed, Zi converges to 0 in probability.