CHAPTER THREE

# transforms and some applications to sums of independent random variables

An understanding of the concepts and applications of transform theory will contribute in several ways to our later work. Transforms are useful for the establishment of valuable general theorems, the determination of moments of random variables, the study of certain probabilistic processes, and the analysis of sums of independent random variables.

Some important applications are introduced in this chapter. However, an appreciation of the power of transform techniques will, for the most part, be developed as we study more advanced topics in later chapters.

## 3-1 The $s$ Transform

Let $f_x(x_0)$ be any PDF. The *exponential transform* (or *s transform*) for this PDF, $f_x{}^T(s)$, is defined by

$$f_x{}^T(s) \equiv E(e^{-sx}) = \int_{-\infty}^{\infty} e^{-sx_0} f_x(x_0)\, dx_0$$

We are interested only in the $s$ transforms of PDF's and not of arbitrary functions. Thus, we need note only those aspects of transform theory which are relevant to this special case.

As long as $f_x(x_0)$ is a PDF, the above integral must be finite at least for the case where $s$ is a pure imaginary quantity (see Prob. 3.01). Furthermore, it can be proved that the $s$ transform of a PDF is unique to that PDF.

Three examples of the calculation of $s$ transforms follow: First, consider the PDF

$$f_x(x_0) = \begin{cases} \lambda e^{-\lambda x_0} & x_0 \geq 0 \\ 0 & x_0 < 0 \end{cases} = \mu_{-1}(x_0 - 0)\lambda e^{-\lambda x_0} \qquad -\infty \leq x_0 \leq \infty$$

$$f_x{}^T(s) = \int_{-\infty}^{\infty} e^{-sx_0} f_x(x_0)\, dx_0 = \int_0^{\infty} \lambda e^{-sx_0} e^{-\lambda x_0}\, dx_0 = \frac{\lambda}{s + \lambda}$$

[The unit step function $\mu_{-1}(x_0 - a)$ is defined in Sec. 2-9.] For a second example, we consider the *uniform* PDF

$$f_x(x_0) = \begin{cases} 1 & \text{if } 0 \leq x_0 \leq 1 \\ 0 & \text{otherwise} \end{cases} = \mu_{-1}(x_0 - 0) - \mu_{-1}(x_0 - 1)$$
$$-\infty \leq x_0 \leq \infty$$

$$f_x{}^T(s) = \int_{-\infty}^{\infty} e^{-sx_0} f_x(x_0)\, dx_0 = \int_0^1 e^{-sx_0}\, dx_0 = \frac{1 - e^{-s}}{s}$$

Our third example establishes a result to be used later. Consider the PDF for a degenerate (deterministic) random variable $x$ which always takes on the experimental value $a$,

$$f_x(x_0) = \mu_0(x_0 - a) \qquad -\infty \leq x_0 \leq \infty$$

$$f_x{}^T(s) = \int_{-\infty}^{\infty} e^{-sx_0} \mu_0(x_0 - a)\, dx_0 = e^{-sa}$$

The PDF corresponding to a given $s$ transform, $f_x{}^T(s)$, is known as the *inverse transform* of $f_x{}^T(s)$. The formal technique for obtaining inverse transforms is beyond the scope of the mathematical prerequisites assumed for this text. For our purposes, we shall often be able to obtain inverse transforms by recognition and by exploiting a few simple properties of transforms. A discussion of one simple procedure for attempting to evaluate inverse transforms will appear in our solution to the example of Sec. 3-8.

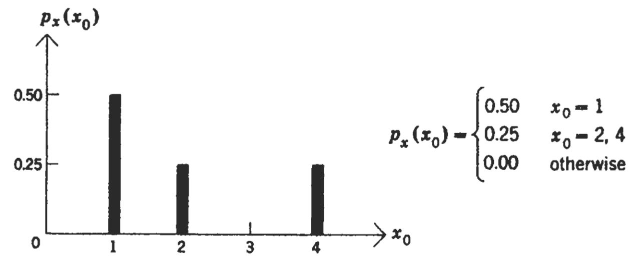


## 3-2 The $z$ Transform

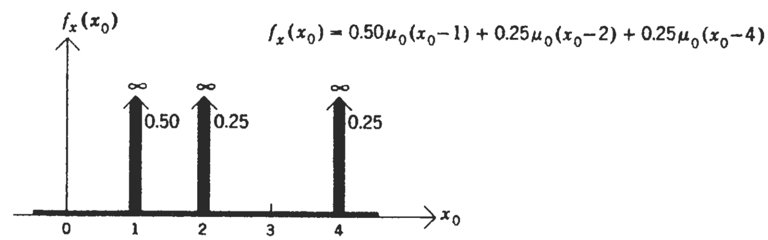
Once we are familiar with the impulse function, any PMF may be expressed as a PDF. To relate the PMF $p_x(x_0)$ to its corresponding PDF $f_x(x_0)$, we use the relation

$$f_x(x_0) = \sum_a p_x(a)\mu_0(x_0 - a)$$

As an example, the PMF $p_x(x_0)$ shown below




may be written as the PDF $f_x(x_0)$,




The $s$ transform of this PDF is obtained from

$$f_x{}^T(s) = \int_{-\infty}^{\infty} e^{-sx_0} f_x(x_0)\, dx_0 = 0.50e^{-s} + 0.25e^{-2s} + 0.25e^{-4s}$$

where we have made use of the following relation from Sec. 2-9

$$\int_{-\infty}^{\infty} \mu_0(x_0 - a) g(x_0)\, dx_0 = g(a)$$

The above $s$ transform could also have been obtained directly from the equivalent (expected value) definition of $f_x{}^T(s)$,

$$f_x^T(s) = E(e^{-sx}) = 0.50e^{-s} + 0.25e^{-2s} + 0.25e^{-4s}$$

Although the $s$ transform is defined for the PDF of *any* random variable, it is convenient to define one additional type of transform for a certain type of PMF. If $p_x(x_0)$ is the PMF for a discrete random variable *which can take on only nonnegative integer experimental values* ($x_0 = 0, 1, 2, \ldots$), we define the *discrete transform* (or *z transform*) of $p_x(x_0)$ to be $p_x^T(z)$, given by

$$p_x^T(z) \equiv E(z^x) = \sum_{x_0=0}^{\infty} z^{x_0} p_x(x_0)$$

We do not find it particularly useful to define a $z$ transform for PMF's which allow noninteger or negative experimental values. In practice, a large number of discrete random variables arise from a count of integer units and from the quantization of a positive quantity, and it is for cases like these that our nonnegative integer constraint holds.

The PMF at the start of this section allows only nonnegative integer values of its random variable. As an example, we obtain the $z$ transform of this PMF,

$$p_x^T(z) = \sum_{x_0=0}^{\infty} z^{x_0} p_x(x_0) = 0.50z + 0.25z^2 + 0.25z^4$$

Note that the $z$ transform for a PMF may be obtained from the $s$ transform of the equivalent PDF by substituting $z = e^{-s}$.

The $z$ transform can be shown to be finite for at least $|z| \le 1$ and to be unique to its PMF. We shall normally go back to a PMF from its transform by recognition of a few familiar transforms. However, we can note from the definition of $p_x^T(z)$,

$$p_x^T(z) = p_x(0) + zp_x(1) + z^2p_x(2) + z^3p_x(3) + \cdots$$

that it is possible to determine the individual terms of the PMF from $p_x^T(z)$ by

$$p_x(x_0) = \frac{1}{x_0!}\left[\frac{d^{x_0}}{dz^{x_0}} p_x^T(z)\right]_{z=0} \qquad x_0 = 0, 1, 2, \ldots$$

## 3-3 Moment-generating Properties of the Transforms

Consider the $n$th derivative with respect to $s$ of $f_x^T(s)$,

$$f_x^T(s) = \int_{x_0=-\infty}^{\infty} e^{-sx_0} f_x(x_0)\,dx_0 \qquad \frac{d^n f_x^T(s)}{ds^n} = \int_{x_0=-\infty}^{\infty} (-x_0)^n e^{-sx_0} f_x(x_0)\,dx_0$$



The right-hand side of the last equation, when evaluated at $s = 0$, may be recognized to be equal to $(-1)^n E(x^n)$. Thus, once we obtain the $s$ transform for a PDF, we can find all the moments by repeated differentiation rather than by performing other integrations.

From the above expression for the $n$th derivative of $f_x^T(s)$, we may establish the following useful results:

$$[f_x^T(s)]_{s=0} = 1 \qquad E(x) = -\left[\frac{df_x^T(s)}{ds}\right]_{s=0} \qquad E(x^2) = \left[\frac{d^2f_x^T(s)}{ds^2}\right]_{s=0}$$

$$\sigma_x^2 = E\{[x - E(x)]^2\} = E(x^2) - [E(x)]^2 = \left\{\frac{d^2f_x^T(s)}{ds^2} - \left[\frac{df_x^T(s)}{ds}\right]^2\right\}_{s=0}$$

Of course, when certain moments of a PDF do not exist, the corresponding derivatives of $f_x^T(s)$ will be infinite when evaluated at $s = 0$.

As one example of the use of these relations, consider the PDF $f_x(x_0) = \mu_{-1}(x_0 - 0)\lambda e^{-\lambda x_0}$, for which we obtained $f_x^T(s) = \lambda/(s + \lambda)$ in Sec. 3-1. We may obtain $E(x)$ and $\sigma_x^2$ by use of the relations

$$E(x) = -\left[\frac{df_x^T(s)}{ds}\right]_{s=0} = -\left[\frac{-\lambda}{(s+\lambda)^2}\right]_{s=0} = \frac{1}{\lambda}$$

$$E(x^2) = (-1)^2\left[\frac{d^2f_x^T(s)}{ds^2}\right]_{s=0} = \left[\frac{2\lambda}{(s+\lambda)^3}\right]_{s=0} = \frac{2}{\lambda^2}$$

$$\sigma_x^2 = E(x^2) - [E(x)]^2 = \frac{1}{\lambda^2}$$

The moments for a PMF may also be obtained by differentiation of its $z$ transform, although the resulting equations are somewhat different from those obtained above. Beginning with the definition of the $z$ transform, we have

$$p_x^T(z) = \sum_{x_0=0}^{\infty} z^{x_0} p_x(x_0)$$

$$\left[\frac{dp_x^T(z)}{dz}\right]_{z=1} = \left[\sum_{x_0=0}^{\infty} x_0 z^{x_0-1} p_x(x_0)\right]_{z=1} = E(x)$$

$$\left[\frac{d^2p_x^T(z)}{dz^2}\right]_{z=1} = \left[\sum_{x_0=0}^{\infty} x_0(x_0-1)z^{x_0-2} p_x(x_0)\right]_{z=1} = E(x^2) - E(x)$$

In general, for $n = 1, 2, \ldots$, we have

$$\frac{d^np_x^T(z)}{dz^n} = \sum_{x_0=0}^{\infty} x_0(x_0-1)(x_0-2)\cdots(x_0-n+1)z^{x_0-n}p_x(x_0)$$

The right-hand side of this last equation, when evaluated at $z = 1$, is equal to some linear combination of $E(x^n)$, $E(x^{n-1})$, . . . , $E(x^2)$, and $E(x)$. What we are accomplishing here is the determination of all moments of a PMF from a single summation (the calculation of the transform itself) rather than attempting to perform a separate summation directly for each moment. This saves quite a bit of work for those PMF's whose $z$ transforms may be obtained in closed form. We shall frequently use the following relations which are easily obtained from the above equations:

$$[p_x^T(z)]_{z=1} = 1 \qquad E(x) = \left[\frac{dp_x^T(z)}{dz}\right]_{z=1} \qquad E(x^2) = \left[\frac{d^2p_x^T(z)}{dz^2} + \frac{dp_x^T(z)}{dz}\right]_{z=1}$$

$$\sigma_x^2 = \left\{\frac{d^2}{dz^2}\,p_x^T(z) + \frac{d}{dz}\,p_x^T(z) - \left[\frac{d}{dz}\,p_x^T(z)\right]^2\right\}_{z=1}$$

We often recognize sums which arise in our work to be similar to expressions for moments of PMF's, and then we may use $z$ transforms to carry out the summations. (Examples of this procedure arise, for instance, in the solutions to Probs. 3.10 and 3.12.)

As an example of the moment-generating properties of the $z$ transform, consider the *geometric* PMF defined by

$$p_x(x_0) = \begin{cases} P(1-P)^{x_0-1} & \text{if } x_0 = 1, 2, 3, \ldots \\ 0 & \text{otherwise} \end{cases} \qquad 0 < P < 1$$

We shall use the $z$ transform to obtain $E(x)$, $E(x^2)$, and $\sigma_x^2$.

$$p_x^T(z) = E(z^x) = \sum_{x_0=0}^{\infty} z^{x_0}p_x(x_0) = \sum_{x_0=1}^{\infty} Pz^{x_0}(1-P)^{x_0-1} = \frac{zP}{1-z(1-P)}$$

After a calculation such as the above we may check $[p_x^T(z)]_{z=1} \overset{?}{=} 1$.

$$E(x) = \left[\frac{d}{dz}\,p_x^T(z)\right]_{z=1} = \frac{1}{P}$$

$$E(x^2) = \left[\frac{d^2}{dz^2}\,p_x^T(z) + \frac{d}{dz}\,p_x^T(z)\right]_{z=1} = \frac{2-P}{P^2}$$

[Try to evaluate $E(x^2)$ directly from the definition of expectation!]

$$\sigma_x^2 = E(x^2) - [E(x)]^2 = \frac{1-P}{P^2}$$

In obtaining $p_x^T(z)$, we used the relation

$$1 + a + a^2 + \cdots + a^k = \frac{1-a^{k+1}}{1-a} \qquad |a| < 1$$



A similar relation which will be used frequently in our work with $z$ transforms is

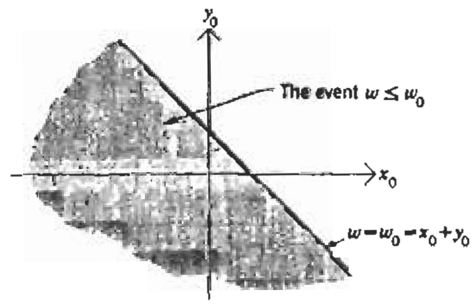
$$1 + a + \frac{a^2}{2!} + \frac{a^3}{3!} + \frac{a^4}{4!} + \cdots = e^a \qquad -\infty < a < \infty$$

### 3-4 Sums of Independent Random Variables; Convolution

The properties of sums of independent random variables is an important topic in the study of probability theory. In this section we approach this topic from a sample-space point of view. A transform approach will be considered in Sec. 3-5, and, in Sec. 3-7, we extend our work to a matter involving the sum of a random number of random variables. Sums of independent random variables also will be our main concern when we discuss limit theorems in Chap. 6.

To begin, we wish to work in an $x_0,y_0$ event space, using the method of Sec. 2-14, to derive the PDF for $w$, the sum of two random variables $x$ and $y$. After a brief look at the general case, we shall specialize our results to the case where $x$ and $y$ are independent.

We are given $f_{x,y}(x_0,y_0)$, the PDF for random variables $x$ and $y$. With $w = x + y$, we go to the $x_0,y_0$ event space to determine $p_{w\le}(w_0)$. The derivative of this CDF is the desired PDF, $f_w(w_0)$.




$$p_{w\le}(w_0) = \int_{x_0=-\infty}^{\infty} dx_0 \int_{y_0=-\infty}^{w_0-x_0} dy_0\, f_{x,y}(x_0,y_0)$$

$$f_w(w_0) = \frac{d}{dw_0}\,p_{w\le}(w_0) = \int_{x_0=-\infty}^{\infty} dx_0\,\frac{d}{dw_0}\left[\int_{y_0=-\infty}^{w_0-x_0} dy_0\, f_{x,y}(x_0,y_0)\right]$$

We may use the formula given in Sec. 2-14 to differentiate the quantity in the brackets to obtain

$$f_w(w_0) = \int_{x_0=-\infty}^{\infty} dx_0\, f_{x,y}(x_0,\, w_0 - x_0)$$

In general, we can proceed no further without specific knowledge of the form of $f_{x,y}(x_0,y_0)$. For the special case where $x$ and $y$ are independent random variables, we may write

$$f_w(w_0) = \int_{x_0=-\infty}^{\infty} dx_0\, f_x(x_0) f_y(w_0 - x_0) \qquad \text{for } w = x + y \text{ and } x,y \text{ independent}$$

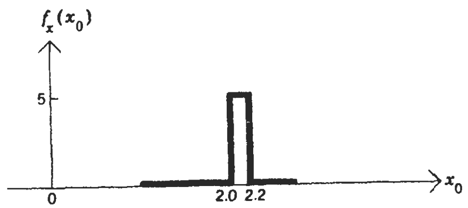
This operation is known as the *convolution* of $f_x(x_0)$ and $f_y(y_0)$. Had we integrated over $x_0$ first instead of $y_0$ in obtaining $p_{w\leq}(w_0)$, we would have found the equivalent expression with $x_0$ and $y_0$ interchanged,

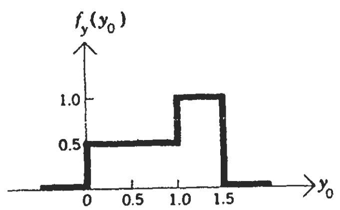
$$f_w(w_0) = \int_{y_0=-\infty}^{\infty} dy_0\, f_y(y_0) f_x(w_0 - y_0)$$

The convolution of two functions has a simple, and often useful, graphical interpretation. If, for instance, we wish to convolve $f_x(x_0)$ and $f_y(y_0)$ using the form

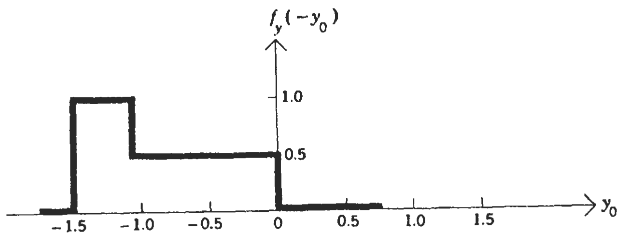
$$f_w(w_0) = \int_{x_0=-\infty}^{\infty} dx_0\, f_x(x_0) f_y(w_0 - x_0)$$

we would require plots of $f_x(x_0)$ and $f_y(w_0 - x_0)$, each plotted as a function of $x_0$. Then, for all possible values of $w_0$, these two curves may be multiplied point by point. The resulting product curve is integrated over $x_0$ to obtain $f_w(w_0)$. Since convolution is often easier to perform than to describe, let's try an example which requires the convolution of the following PDF's:



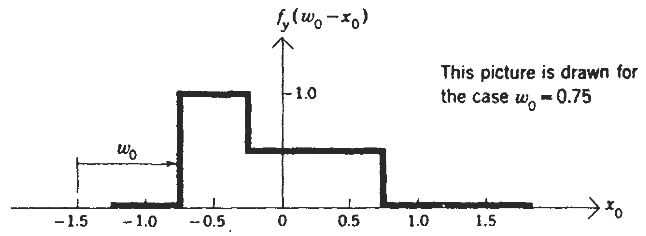




We are asked to find the PDF for $w = x + y$, given that $x$ and $y$ are independent random variables. To obtain the desired plot of $f_y(w_0 - x_0)$ as a function of $x_0$, we first "flip" $f_y(y_0)$ about the line $y_0 = 0$ to get

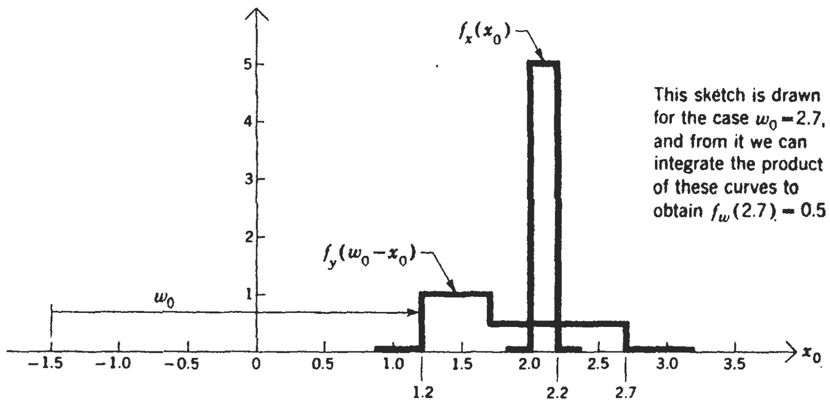




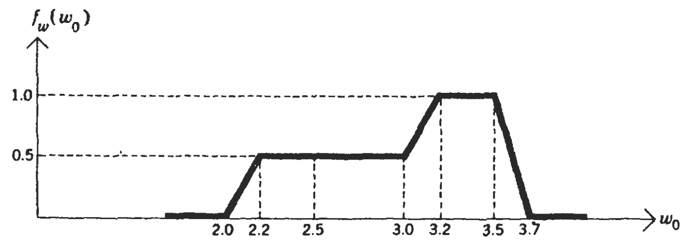

We next replace variable $y_0$ by $x_0$ and then plot, along an $x_0$ axis, the flipped function shifted to the right by $w_0$.




We can now present $f_y(w_0 - x_0)$ on the same plot as $f_x(x_0)$ and perform the integration of the product of the curves as a function of $w_0$.




Our final step is to plot the integral of the product of these two functions for all values of $w_0$. We obtain




Thus, by graphical convolution, we have determined the PDF for random variable $w$, which, you may recall, was defined to be the sum of independent random variables $x$ and $y$. Of course, we check to see that $f_w(w_0)$ is nonzero only in the range of possible values for sums of $x$ and $y$ ($2.0 \leq w_0 \leq 3.7$) and that this derived PDF integrates to unity.

We are now familiar with two equivalent methods to obtain the PDF for the sum of the *independent* random variables $x$ and $y$. One method would be to work directly in the $x_0,y_0$ event space; an alternative is to perform the convolution of their marginal PDF's. In the next section, a transform technique for this problem will be introduced.

For the special case where we are to convolve two PDF's, *each* of which contains one or more impulses, a further note is required. Our simplified definition of the impulse does not allow us to argue the following result from that definition; so we shall simply define the integral of the product of two impulses to be

$$\int_{x_0=-\infty}^{\infty} \mu_0(x_0 - a)\mu_0(x_0 - b)\, dx_0 = \mu_0(a - b)$$

Thus, the convolution of two impulses would be another impulse, with an area equal to the product of the areas of the two impulses.

A special case of convolution, the *discrete* convolution, is introduced in Prob. 3.17. The discrete convolution allows one to convolve PMF's directly without first replacing them by their equivalent PDF's.

## 3-5 The Transform of the PDF for the Sum of Independent Random Variables

Let $w = x + y$, where $x$ and $y$ are independent random variables with marginal PDF's $f_x(x_0)$ and $f_y(y_0)$. We shall obtain $f_w{}^T(s)$, the $s$ transform of $f_w(w_0)$, from the transforms $f_x{}^T(s)$ and $f_y{}^T(s)$.

$$f_w{}^T(s) = E(e^{-sw}) = E(e^{-s(x+y)})$$
$$= \int_{x_0=-\infty}^{\infty} \int_{y_0=-\infty}^{\infty} e^{-sx_0}e^{-sy_0} f_{x,y}(x_0,y_0)\, dx_0\, dy_0$$

The compound PDF factors into $f_x(x_0)f_y(y_0)$ because of the independence of $x$ and $y$, to yield

$$f_w{}^T(s) = \int_{x_0=-\infty}^{\infty} e^{-sx_0} f_x(x_0)\, dx_0 \int_{y_0=-\infty}^{\infty} e^{-sy_0} f_y(y_0)\, dy_0$$

$$f_w{}^T(s) = f_x{}^T(s) f_y{}^T(s) \qquad \text{for } w = x + y \text{ and } x,y \text{ statistically independent}$$

We have proved that the transform of the PDF of a random variable which is the sum of two *independent* random variables is the product of the transforms of their PDF's.

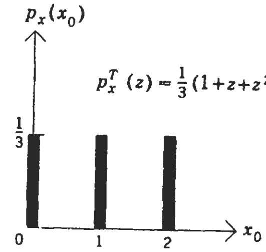
The proof of the equivalent result for discrete random variables which have $z$ transforms,

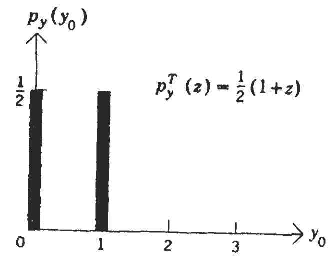


$$p_w{}^T(z) = p_x{}^T(z) p_y{}^T(z) \qquad \text{for } w = x + y \text{ and } x,y \text{ statistically independent}$$

is entirely similar to the above.

Let's do one example for the discrete case, using $z$ transforms. Independent variables $x$ and $y$ are described by the PMF's

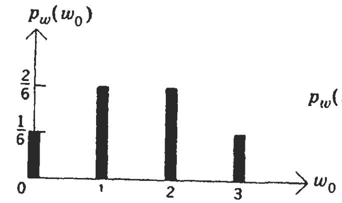



The PMF for random variable $w$, given that $w = x + y$, has the $z$ transform

$$p_w{}^T(z) = p_x{}^T(z) p_y{}^T(z) = \tfrac{1}{6}(1 + 2z + 2z^2 + z^3)$$

and since we know $p_w{}^T(z) = \sum_{w_0=0}^{\infty} p_w(w_0) z^{w_0}$, we can note that the coefficient of $z^{w_0}$ in $p_w{}^T(z)$ is equal to $p_w(w_0)$. Thus, we may take the inverse transform of $p_w{}^T(z)$ to obtain




$$p_w(w_0) = \begin{cases} \frac{1}{6} & w_0 = 0, 3 \\ \frac{2}{6} & w_0 = 1, 2 \\ 0 & \text{otherwise} \end{cases}$$

The reader is encouraged to either convolve the PMF's or work the problem in an $x_0,y_0$ sample space to verify the above result.

## 3-6 A Further Note on Sums of Independent Random Variables

For *any* random variables $x$ and $y$, we proved

$$E(x + y) = E(x) + E(y)$$

in Sec. 2-7. Thus, the expected value of a sum is always equal to the sum of the expected values of the individual terms.

We next wish to note how variances combine when we add *independent* random variables to obtain new random variables. Let $w = x + y$; we then have, using the above relation for the expected value of a sum,

$$\sigma_w^2 = E\{[w - E(w)]^2\} = E\{[x + y - E(x) - E(y)]^2\}$$
$$= E\{[x - E(x) + y - E(y)]^2\}$$
$$= E\{[x - E(x)]^2\} + E\{[y - E(y)]^2\} + 2E\{[x - E(x)][y - E(y)]\}$$
$$= \sigma_x^2 + \sigma_y^2 + 2E[xy - xE(y) - yE(x) + E(x)E(y)]$$

For $x$ and $y$ independent, the expected values of all products in the last brackets are equal. In fact, only *linear* independence is required for this to be true and we obtain the following important expression for the variance of the sum of linearly independent random variables:

$$\sigma_w^2 = \sigma_x^2 + \sigma_y^2 \qquad \text{for } w = x + y \text{ and } \quad x,y \text{ linearly independent}$$

An alternative derivation of this relation for statistically independent random variables (using transforms) is indicated in Prob. 3.14.

We now specialize our work to sums of independent random variables for the case where each member of the sum has the same PDF. When we are concerned with this case, which may be considered as a sum of independent experimental values from an experiment whose outcome is described by a particular PDF, we speak of *independent identically distributed* random variables.

Let $r$ be the sum of $n$ independent identically distributed random variables, each with expected value $E(x)$ and variance $\sigma_x^2$. We already know that

$$E(r) = nE(x) \qquad \sigma_r^2 = n\sigma_x^2 \qquad \sigma_r = \sqrt{n}\,\sigma_x$$

For the rest of this section *we consider only the case* $E(x) > 0$ *and* $\infty > \sigma_x^2 > 0$. The $E(x) > 0$ condition will simplify our statements and expressions. Our reasoning need not hold for $\sigma_x^2 = \infty$, and any PDF which has $\sigma_x^2 = 0$ represents an uninteresting deterministic quantity.

If we are willing to accept the standard deviation of $r$, $\sigma_r$, as a type of linear measure of the spread of a PDF about its mean, some interesting *speculations* follow.

As $n$ increases, the PDF for $r$ gets "wider" (as $\sqrt{n}$) and its expected value increases (as $n$). The mean and the standard deviation both grow, but the mean increases more rapidly.



This would lead us to expect that, for instance, as $n$ goes to infinity, the probability that an experimental value of $r$ falls within a certain absolute distance $d$ of $E(r)$ decreases to zero. That is,

$$\lim_{n\to\infty} \text{Prob}[|r - E(r)| < d] = 0 \qquad \text{(we think)}$$

We reach this speculation by reasoning that, as the width of $f_r(r_0)$ grows as $\sqrt{n}$, the height of most of the curve should fall as $1/\sqrt{n}$ to keep its area equal to unity. If so, the area of $f_r(r_0)$ over a slit of fixed width $2d$ should go to zero.

Our second speculation is based on the fact that $E(r)$ grows faster than $\sigma_r$. We might then expect that the probability that an experimental value of $r$ falls within $\pm A\,\%$ of $E(r)$ grows to unity as $n$ goes to infinity (for $A \neq 0$). That is,

$$\lim_{n\to\infty} \text{Prob}\left[\frac{|r - E(r)|}{E(r)} < \frac{A}{100}\right] = 1 \qquad \text{for } A > 0 \qquad \text{(we think)}$$

We might reason that, while the height of the PDF in most places near $E(r)$ is probably falling as $1/\sqrt{n}$, the interval of interest, defined by

$$|r - E(r)| < \frac{A}{100} E(r)$$

grows as $n$. Thus the area over this interval should, as $n \to \infty$, come to include all the area of the PDF $f_r(r_0)$.

For the given conditions, we shall learn in Chap. 6 that these speculations happen to be correct. Although proofs of such theorems could be stated here, we would not have as satisfactory a physical interpretation of such *limit theorems* as is possible after we become familiar with the properties of several important PMF's and PDF's.

## 3-7 Sum of a Random Number of Independent Identically Distributed Random Variables

Let $x$ be a random variable with PDF $f_x(x_0)$ and $s$ transform $f_x^T(s)$. If $r$ is defined to be the sum of $n$ independent experimental values of random variable $x$, we know from the results of Sec. 3-5 that the transform for the PDF $f_r(r_0)$ is

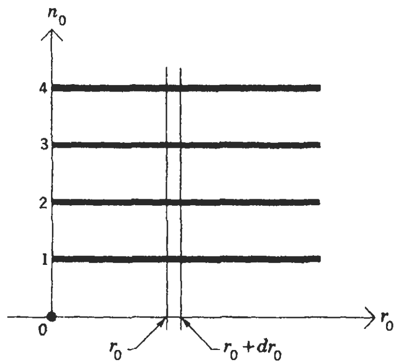
$f_r^T(s) = [f_x^T(s)]^n$ where $r$ is the sum of $n$ (statistically) independent experimental values of $x$

We now wish to consider the situation when $n$ is also a random variable [with PMF $p_n(n_0)$]. We are interested in the sum of a random (but integer) number of independent identically distributed random variables.

For instance, if $f_x(x_0)$ were the PDF for the weight of any individual in an elevator, if the weights of people in the elevator could be considered to be independent random variables, *and* if $p_n(n_0)$ were the PMF for the number of people in the elevator, random variable $r$ would represent the total weight of the people in the elevator. *Our work will also require that $x$ and $n$ be independent.* In our example, the PDF for the individual weights may not depend on the number of people in the elevator.

If $n$ can take on the experimental values 0, 1, 2, . . . , $N$, the *sample* space for each performance of our experiment is of $N + 1$ dimensions, since each performance generates one experimental value of $n$ and up to $N$ experimental values of random variable $x$. It is usually difficult to obtain the desired PDF $f_r(r_0)$ directly, but its $s$ transform is derived quite easily. Although it may be difficult to get back to $f_r(r_0)$ in a useful form from $f_r^T(s)$, it is a simple matter to evaluate the moments and variance of random variable $r$.

We may determine the $s$ transform for $f_r(r_0)$ by working in an event space for random variable $n$ (the number of independent experimental values of $x$ in the sum) and $r$ (the value of the sum). This event space, perhaps a strange choice at first sight, consists of a set of parallel lines in one quadrant and one point at the origin of the $r_0,n_0$ plane.




Every event point representing a possible outcome is either on one of these heavy lines or at the origin

Along each heavy line, there applies a conditional PDF $f_{r|n}(r_0 \mid n_0)$ which is the PDF for $r$ given the experimental value of $n$. We know that $f_{r|n}(r_0 \mid n_0)$ is that PDF which describes the sum of $n_0$ independent experimental values of random variable $x$. As we noted at the start of this section, PDF $f_{r|n}(r_0 \mid n_0)$ has the $s$ transform $[f_x^T(s)]^{n_0}$.

We use these observations to collect $f_r(r_0)$ as a summation in the



$r_0,n_0$ event space, and then we take the $s$ transform on both sides of the equation.

$$f_r(r_0) = \sum_{n_0} p_n(n_0) f_{r|n}(r_0 \mid n_0)$$

$$f_r^T(s) = \int_{-\infty}^{\infty} e^{-sr_0} \sum_{n_0} p_n(n_0) f_{r|n}(r_0 \mid n_0)\, dr_0$$

$$= \sum_{n_0} p_n(n_0) \int_{-\infty}^{\infty} e^{-sr_0} f_{r|n}(r_0 \mid n_0)\, dr_0$$

$$= \sum_{n_0} p_n(n_0) [f_x^T(s)]^{n_0}$$

We recognize the last equation for $f_r^T(s)$ to be the $z$ transform of PMF $p_n(n_0)$, with the transform evaluated at $z = f_x^T(s)$. We now restate this problem and its solution.

*Let $n$ and $x$ be independent random variables, where $n$ is described by the PMF $p_n(n_0)$ and $x$ by the PDF $f_x(x_0)$. Define $r$ to be the sum of $n$ independent experimental values of random variable $x$. The $s$ transform for the PDF $f_r(r_0)$ is* $f_r^T(s) = p_n^T[f_x^T(s)]$

We may use the chain rule for differentiation to obtain the expectation, second moment, and variance of $r$.

$$E(r) = -\left[\frac{df_r^T(s)}{ds}\right]_{s=0} = -\left\{\frac{dp_n^T[f_x^T(s)]}{d[f_x^T(s)]} \cdot \frac{df_x^T(s)}{ds}\right\}_{s=0}$$

To evaluate the first term in the right-hand brackets, we proceed,

$$\left\{\frac{dp_n^T[f_x^T(s)]}{d[f_x^T(s)]}\right\}_{s=0} = \left[\frac{dp_n^T(z)}{dz}\right]_{z=1} = E(n)$$

The first step in the equation immediately above made use of the fact that $[f_x^T(s)]_{s=0} = 1$. To solve for $E(r)$, we use $\left[\frac{df_x^T(s)}{ds}\right]_{s=0} = -E(x)$ in the expression for $E(r)$ to obtain

$$E(r) = E(n)E(x)$$

A second chain-rule differentiation of $f_r^T(s)$ and the use of the relation for $\sigma_r^2$ in terms of $f_r^T(s)$ leads to the further result

$$\sigma_r^2 = E(n)\sigma_x^2 + [E(x)]^2\sigma_n^2$$

We may note that this result checks out correctly for the case where $n$ is deterministic ($\sigma_n^2 = 0$) and for the case where $x$ is deterministic ($\sigma_x^2 = 0$, $[E(x)]^2 = x^2$).

If we had required that $x$, as well as $n$, be a discrete random variable which takes on only nonnegative integer experimental values, we could have worked with the PMF $p_x(x_0)$ to study a particular case of the above derivations. The resulting $z$ transform of the PMF for discrete random variable $r$ is

$$p_r^T(z) = p_n^T[p_x^T(z)]$$

and the above expressions for $E(r)$ and $\sigma_r^2$ still hold.

An example of the sum of a random number of independent identically distributed random variables is included in the following section.

## 3-8 An Example, with Some Notes on Inverse Transforms

As we undertake the study of some common probabilistic processes in the next chapter, our work will include numerous examples of applications of transform techniques. One problem is solved here to review some of the things we learned in this chapter, to indicate one new application, and [in part (e)] to lead us into a discussion of how we may attempt to go back from an $s$ transform to its PDF. There are, of course, more general methods, which we shall not discuss.

Let discrete random variable $k$ be described by the PMF

$$p_k(k_0) = \frac{8^{k_0}}{9^{k_0+1}} \qquad k_0 = 0, 1, 2, \ldots$$

(a) Determine the expected value and variance of random variable $k$.
(b) Determine the probability that an experimental value of $k$ is even.
(c) Determine the probability that the sum of $n$ independent experimental values of $k$ is even.
(d) Let random variable $k$ represent the number of light bulbs we happen to have on hand at time $T_0$. Furthermore, let $x$, the lifetime of each bulb, be an independent random variable with PDF

$$f_x(x_0) = \mu_{-1}(x_0 - 0)\lambda e^{-\lambda x_0}$$

We turn on one bulb at time $T_0$, replacing it immediately with another bulb as soon as it fails. This continues until the last of the $k$ bulbs blows out. Determine the $s$ transform, expectation, and variance for random variable $r$, the time from $T_0$ until the last bulb dies.



(e) Determine the PDF $f_r(r_0)$ from the $s$ transform $f_r^T(s)$ obtained in part (d).

**a** Rather than attempt to carry out some troublesome summations directly, it seems appropriate to employ the $z$ transform.

$$p_k^T(z) = E(z^k) = \sum_{k_0=0}^{\infty} z^{k_0}p_k(k_0) = \frac{1}{9}\sum_{k_0=0}^{\infty}\left(\frac{8z}{9}\right)^{k_0} = (9 - 8z)^{-1}$$

Making a quick check, we note that $p_k^T(1)$ is equal to unity.

$$E(k) = \left[\frac{d}{dz}p_k^T(z)\right]_{z=1} = 8$$

$$\sigma_k^2 = \left[\frac{d^2}{dz^2}p_k^T(z) + \frac{d}{dz}p_k^T(z) - \left(\frac{d}{dz}p_k^T(z)\right)^2\right]_{z=1} = 72$$

**b** We shall do this part two ways. First, for our particular PMF we can evaluate the answer directly. Let $A$ represent the event that the experimental value of $k$ is even.

$$P(A) = \sum_{k_0 \text{ even}} p_k(k_0) = \frac{1}{9}\left(1 + \frac{8^2}{9^2} + \frac{8^4}{9^4} + \frac{8^6}{9^6} + \cdots\right)$$

$$P(A) = \frac{1/9}{1 - 64/81} = \frac{9}{17}$$

The monotonically decreasing PMF for random variable $k$ makes it obvious that $P(A) > 0.5$, since $p_k(0) > p_k(1)$, $p_k(2) > p_k(3)$, etc.

Another approach, which is applicable to a more general problem where we may not be able to sum $\sum_{k_0 \text{ even}} p_k(k_0)$ directly, follows:

$$P(A) = \sum_{k_0 \text{ even}} p_k(k_0) = \frac{1}{2}\left[\sum_{k_0} p_k(k_0)(1)^{k_0} + \sum_{k_0} p_k(k_0)(-1)^{k_0}\right]$$

$$P(A) = \tfrac{1}{2}[1 + p_k^T(-1)]$$

For our example we have $p_k^T(z) = (9 - 8z)^{-1}$, $p_k^T(-1) = \frac{1}{17}$, resulting in $P(A) = 9/17$.

**c** Let $r$ be the sum of $n$ independent experimental values of random variable $k$. In Sec. 3-5, we learned that

$$p_r^T(z) = [p_k^T(z)]^n$$

which we simply substitute into the expression obtained in (b) above, to get

$$\text{Prob(exper. value of } r \text{ is even)} = \tfrac{1}{2}[1 + p_r^T(-1)] = \tfrac{1}{2}[1 + (\tfrac{1}{17})^n]$$

As we might expect on intuitive grounds, this probability rapidly approaches 0.5 as $n$ grows.

**d** This part is concerned with the sum of a random number of independent identically distributed random variables. Continuous random variable $\tau$ is the sum of $k$ independent experimental values of random variable $x$. From Sec. 3-7, we have

$$f_\tau^T(s) = p_k^T[f_x^T(s)] = [p_k^T(z)]_{z=f_x^T(s)}$$

For $f_x(x_0)$, the exponential PDF, we have

$$f_x^T(s) = \int_{x_0=0}^{\infty} e^{-sx_0}\lambda e^{-\lambda x_0}\,dx_0 = \frac{\lambda}{s+\lambda}$$

which results in

$$f_\tau^T(s) = \left(9 - \frac{8\lambda}{s+\lambda}\right)^{-1} = \frac{s+\lambda}{9s+\lambda} \qquad [f_\tau^T(s)]_{s=0} \stackrel{\vee}{=} 1$$

We may substitute $E(k)$, $\sigma_k^2$, $E(x)$, and $\sigma_x^2$ into the formulas of Sec. 3-7 to obtain $E(\tau)$ and $\sigma_\tau^2$, or we may use the relations

$$E(\tau) = -\left[\frac{d}{ds}\left(\frac{s+\lambda}{9s+\lambda}\right)\right]_{s=0} \quad \text{and} \quad \sigma_\tau^2 = \left[\frac{d^2}{ds^2}\left(\frac{s+\lambda}{9s+\lambda}\right)\right]_{s=0} - \left[\frac{d}{ds}\left(\frac{s+\lambda}{9s+\lambda}\right)\right]_{s=0}^2$$

We'll use the former method, with

$$E(k) = 8$$

$$\sigma_k^2 = 72$$

from part (a) of this example and

$$E(x) = \frac{1}{\lambda}$$

$$\sigma_x^2 = \frac{1}{\lambda^2}$$

(from the example in Sec. 3-3), which, in the expressions of Sec. 3-7 for the expectation and variance of a sum of a random number of independent identically distributed random variables, yields

$$E(\tau) = E(k)E(x) = \frac{8}{\lambda} \qquad \sigma_\tau^2 = E(k)\sigma_x^2 + [E(x)]^2\sigma_k^2 = \frac{80}{\lambda^2}$$

The expected time until the last bulb dies is the same as it would be if we always had eight bulbs. But, because of the probabilistic behavior of $k$, the variance of this time is far greater than the value $8/\lambda^2$ which would describe $\sigma_\tau^2$ if we always started out with exactly eight bulbs.

**e** Let $A_1, A_2, \ldots, A_k$ be a list of mutually exclusive collectively exhaustive events. Assume that there is a continuous random variable $y$ which is not independent of the $A_i$'s. Then it is useful to write

$$f_y(y_0) = \sum_i P(A_i)f_{y|A_i}(y_0 \mid A_i)$$



And, from the definition of the $s$ transform, we note that $f_y^T(s)$ would be the weighted sum of the transforms of the conditional PDF's $f_{y|A_i}(y_0 \mid A_i)$. If we define

$$f_{y|A_i}^T(s) = \int_{y_0=-\infty}^{\infty} e^{-sy_0}f_{y|A_i}(y_0 \mid A_i)\,dy_0 = E(e^{-sy} \mid A_i)$$

we have

$$f_y^T(s) = \sum_i P(A_i)f_{y|A_i}^T(s)$$

When we wish to take inverse transforms (go back to a PDF from a transform), we shall try to express the $s$ transform to be inverted, $f_y^T(s)$, in the above form such that we can recognize the inverse transform of each $f_{y|A_i}^T(s)$.

In our particular problem, where the PMF for $k$ is of the form

$$p_k(k_0) = (1-P)P^{k_0} \qquad k_0 = 0, 1\ 2, \ldots; \quad 1 > P > 0$$

and the PDF for $x$ is the exponential PDF

$$f_x(x_0) = \lambda\mu_{-1}(x_0 - 0)e^{-\lambda x_0}$$

it happens that we may obtain $f_\tau(\tau_0)$ from $f_\tau^T(s)$ by the procedure discussed above. We begin by using long division to obtain

$$f_\tau^T(s) = \frac{s+\lambda}{9s+\lambda} = \frac{1}{9} + \frac{8}{9}\,\frac{\lambda/9}{s+\lambda/9}$$

which is of the form

$$f_\tau^T(s) = \tfrac{1}{9}f_{\tau|A_1}^T(s) + \tfrac{8}{9}f_{\tau|A_2}^T(s)$$

From the examples carried out in Sec. 3-1, we note that

$$f_{\tau|A_1}^T(s) = 1$$

has the inverse transform $f_{\tau|A_1}(\tau_0 \mid A_1) = \mu_0(\tau_0 - 0)$, and also

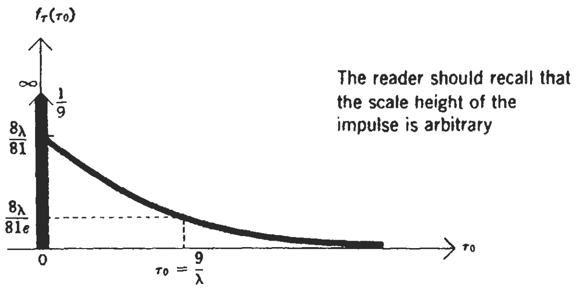
$$f_{\tau|A_2}^T(s) = \frac{\lambda/9}{s+\lambda/9}$$

has the inverse transform $f_{\tau|A_2}(\tau_0 \mid A_2) = \mu_{-1}(\tau_0 - 0)\,\frac{\lambda}{9}\,e^{-\lambda\tau_0/9}$

and, finally, we have the PDF for the duration of the interval during which the lights are on.

$$f_\tau(\tau_0) = \tfrac{1}{9}\mu_0(\tau_0 - 0) + \tfrac{8}{9}\mu_{-1}(\tau_0 - 0)\,\frac{\lambda}{9}\,e^{-\lambda\tau_0/9}$$

The impulse at $\tau_0 = 0$ in this PDF is due to the fact that, with probability $\frac{1}{9}$, we start out with zero bulbs. Thus our PDF $f_\tau(\tau_0)$ is a *mixed* PDF, having both a discrete and a continuous component. We conclude with a sketch of the PDF $f_\tau(\tau_0)$




### 3-9 Where Do We Go from Here?

We are already familiar with most of the basic concepts, methods, and tools of applied probability theory. By always reasoning in an appropriate sample or event space, we have had little difficulty in going from concepts to applications.

Three main areas of study are to follow:

1 Probabilistic processes (Chaps. 4 and 5)
2 Limit theorems (Chap. 6)
3 Statistical reasoning (Chap. 7)

Although we shall consider these topics in the above order, this does not necessarily reflect their relative importance in the world of applied probability theory. Further study of the consequences of the summation of a large number of random variables (limit theorems) is indeed basic. Many probabilicists work solely at attempting to make reasonable inferences from actual physical data (statistical reasoning).

Our choice of the order of these topics is based on the contention that, if we first develop an understanding of several processes and their properties, we may then begin a more meaningful discussion of limit theorems and statistics. The following two chapters are concerned with those probabilistic processes which form the most basic building blocks from which models of actual physical processes are constructed.



## PROBLEMS

**3.01** A sufficient condition for the existence of an integral is that the integral of the *magnitude* of the integrand be finite. Show that, at least for purely imaginary values of $s$, $s = j\omega$, this condition is always satisfied by the $s$ transform of a PDF.

**3.02** If we allow $s$ to be the complex quantity, $s = \alpha + j\omega$, determine for which values of $s$ in the $\alpha,\omega$ plane the $s$ transforms of the following PDF's exist:

**a** $f_x(x_0) = \begin{cases} \lambda e^{-\lambda x_0} & x_0 \geq 0 \\ 0 & x_0 < 0 \end{cases}$ **b** $f_x(x_0) = \begin{cases} 0 & x_0 > 0 \\ \lambda e^{\lambda x_0} & x_0 \leq 0 \end{cases}$

**c** $f_x(x_0) = \begin{cases} 0.5\lambda e^{-\lambda x_0} & x_0 \geq 0 \\ 0.5\lambda e^{\lambda x_0} & x_0 < 0 \end{cases}$

**3.03** Express the PMF $p_x(x_0) = (1 - P)P^{x_0}$, $x_0 = 0, 1, 2, \ldots$ , as a PDF.

**3.04** If $z$ can be the complex number $z = \alpha + j\beta$, determine for which values of $z$ in the $\alpha,\beta$ plane the $z$ transform of the PMF of Prob. 3.03 will exist.

**3.05** All parts of this problem require numerical answers.
**a** If $f_y{}^T(s) = K/(2 + s)$, evaluate $K$ and $E(y^3)$.
**b** If $p_x{}^T(z) = (1 + z^2)/2$, evaluate $p_x[E(x)]$ and $\sigma_x$.
**c** If $f_x{}^T(s) = 2(2 - e^{-s/2} - e^{-s})/3s$, evaluate $E(e^{2x})$.
**d** If $p_x{}^T(z) = A(1 + 3z)^3$, evaluate $E(x^3)$ and $p_x(2)$.

**3.06** Determine whether or not the following are valid $z$ transforms of a PMF for a discrete random variable which can take on only nonnegative integer experimental values:
**a** $z^2 + 2z - 2$ **b** $2 - z$ **c** $(2 - z)^{-1}$

**3.07** Show that neither of the following is an $s$ transform of a PDF:
**a** $(1 - e^{-5s})/s$ **b** $7(4 + 3s)^{-1}$

**3.08** Let $l$ be a discrete random variable whose possible experimental values are all nonnegative integers. We are given

$$p_l{}^T(z) = K\left[\frac{14 + 5z - 3z^2}{8(2 - z)}\right]$$

Determine the numerical values of $E(l)$, $p_l(1)$ and of the conditional expected value of $l$ given $l \neq 0$.

**3.09** Use the expected-value definition of the $s$ transform to prove that, if $x$ and $y$ are random variables with $y = ax + b$, $f_y^T(s) = e^{-sb}f_x^T(as)$. (This is a useful expression, and we shall use it in our proof of the central limit theorem in Chap. 6.)

**3.10** For a particular batch of cookies, $k$, the number of nuts in any cookie is an independent random variable described by the probability mass function

$$p_k(k_0) = \tfrac{1}{3}(\tfrac{2}{3})^{k_0} \qquad k_0 = 0, 1, 2, 3, \ldots$$

Human tastes being what they are, assume that the cash value of a cookie is proportional to the third power of the number of nuts in the cookie. The cookie packers (they are chimpanzees) eat all the cookies containing exactly 0, 1, or 2 nuts. All series must be summed.

**a** What is the probability that a randomly selected cookie is eaten by the chimpanzees?

**b** What is the probability that a particular nut, chosen at random from the population of all nuts, is eaten by the chimpanzees?

**c** What is the fraction of the cash value which the chimpanzees consume?

**d** What is the probability that a random nut will go into a cookie containing exactly $R$ nuts?

**3.11** The hitherto uncaught burglar is hiding in city $A$ (with a priori probability 0.3) or in city $B$ (with a priori probability 0.6), or he has left the country. If he is in city $A$ and $N_A$ men are assigned to look for him there, he will be caught with probability $1 - f^{N_A}$. If he is in city $B$ and $N_B$ men are assigned to look for him there, he will be caught with probability $1 - f^{N_B}$. If he has left the country, he won't be captured.

Policemen's lives being as hectic as they are, $N_A$ and $N_B$ are independent random variables described by the probability mass functions

$$p_{N_A}(N) = \frac{2^N e^{-2}}{N!} \qquad N = 0, 1, 2, \ldots$$

$$p_{N_B}(N) = (\tfrac{1}{2})^N \qquad N = 1, 2, 3, \ldots$$

**a** What is the probability that a total of three men will be assigned to search for the burglar?

**b** What is the probability that the burglar will be caught? (All series are to be summed.)

**c** Given that he was captured in a city in which exactly $K$ men had been assigned to look for him, what is the probability that he was found in city $A$?



**3.12** The number of "leads" (contacts) available to a salesman on any given day is a Poisson random variable with probability mass function

$$p_k(k_0) = \frac{\mu^{k_0} e^{-\mu}}{k_0!} \qquad k_0 = 0, 1, 2, \ldots$$

The probability that any particular lead will result in a sale is 0.5. If your answers contain any series, the series must be summed.

**a** What is the probability that the salesman will make exactly one sale on any given day?

**b** If we randomly select a sales receipt from his file, what is the probability that it represents a sale made on a day when he had a total of $R$ leads?

**c** What fraction of all his leads comes on days when he has exactly one sale?

**d** What is the probability that he has no sales on a given day?

**3.13** The probability that a store will have exactly $k_0$ customers on any given day is

$$p_k(k_0) = \tfrac{1}{5}(\tfrac{4}{5})^{k_0} \qquad k_0 = 0, 1, 2, \ldots$$

On each day when the store has had at least one customer, one of the sales slips for that day is picked out of a hat, and a door prize is mailed to the corresponding customer. (No customer goes to this store more than once or buys more or less than exactly one item.)

**a** What is the probability that a customer selected randomly from the population of all customers will win a door prize?

**b** Given a customer who has won a door prize, what is the probability that he was in the store on a day when it had a total of exactly $k_0$ customers?

**3.14** Independent random variables $x$ and $y$ have PDF's whose $s$ transforms are $f_x^T(s)$ and $f_y^T(s)$. Random variable $r$ is defined to be $r = x + y$. Use $f_r^T(s)$ and the moment generating properties of transforms to show that $E(r) = E(x) + E(y)$ and $\sigma_r^2 = \sigma_x^2 + \sigma_y^2$.

**3.15** Let $x$ and $y$ be independent random variables with

$$f_x(x_0) = \begin{cases} \lambda e^{-\lambda x_0} & x_0 \geq 0 \\ 0 & x_0 < 0 \end{cases} \qquad f_y(y_0) = \begin{cases} 0 & y_0 > 0 \\ \lambda e^{\lambda y_0} & y_0 \leq 0 \end{cases}$$

Random variable $r$ is defined by $r = x + y$.

Determine:

**a** $f_x^T(s)$, $f_y^T(s)$, and $f_r^T(s)$.

**b** $E(r)$ and $\sigma_r^2$.
**c** $f_r(r_0)$.
**d** Repeat the previous parts for the case $r = ax + by$.

**3.16** Consider the PDF $f_x(x_0) = \mu_{-1}(x_0 - 0) - \mu_{-1}(x_0 - 1)$. Random variable $y$ is defined to be the sum of two independent experimental values of $x$. Determine the PDF $f_y(y_0)$:
**a** In an appropriate two-dimensional event space
**b** By performing the convolution graphically
**c** By taking the inverse transform of $f_y^T(s)$ (if you can)

**3.17** **a** If $x$ and $y$ are *any* independent discrete random variables with PMF's $p_x(x_0)$ and $p_y(y_0)$ and we define $r = x + y$, show that $p_r(r_0) = \sum_{x_0} p_x(x_0)p_y(r_0 - x_0) = \sum_{y_0} p_y(y_0)p_x(r_0 - y_0)$. These summations are said to represent the *discrete convolution*. Show how you would go about performing a discrete convolution graphically.
**b** For the case where $x$ and $y$ are discrete, independent random variables which can take on only nonnegative-integer experimental values, take the $z$ transform of one of the above expressions for $p_r(r_0)$ to show that $p_r^T(z) = p_x^T(z)p_y^T(z)$.

**3.18** Random variable $x$ has the PDF $f_x(x_0)$, and we define the *Mellin transform* $f_x^M(s)$ to be

$$f_x^M(s) = E(x^s)$$

**a** Determine $E(x)$ and $\sigma_x^2$ in terms of $f_x^M(s)$.
**b** Let $y$ be a random variable with

$$f_y(y_0) = Ky_0f_x(y_0)$$

**i** Determine $K$.
**ii** Determine $f_y^M(s)$ in terms of $f_x^M(s)$.
**iii** Evaluate $f_x^M(s)$ and $f_y^M(s)$ for

$$f_x(x_0) = \begin{cases} 1 & \text{if } 0 < x_0 \le 1 \\ 0 & \text{otherwise} \end{cases}$$

and use your results to determine $E(y)$ and $\sigma_y$.
**c** Let $w$ and $r$ be independent random variables with PDF's $f_w(w_0)$ and $f_r(r_0)$ and Mellin transforms $f_w^M(s)$ and $f_r^M(s)$. If we define $l = wr$, find $f_l^M(s)$ in terms of the Mellin transforms for $w$ and $r$.

**3.19** A fair wheel of fortune, calibrated infinitely finely from zero to unity, is spun $k$ times, and the resulting readings are summed to obtain an experimental value of random variable $r$. Discrete random variable $k$ has the PMF $p_k(k_0) = \frac{\lambda^{k_0}e^{-\lambda}}{k_0!}$, $k_0 = 0, 1, 2, \ldots$.



Determine:
**a** The probability that at least one reading is larger than 0.3
**b** $f_r^T(s)$
**c** $E(r^2)$

**3.20** Widgets are packed into cartons which are packed into crates. The weight (in pounds) of a widget is a continuous random variable with PDF

$$f_x(x_0) = \lambda e^{-\lambda x_0} \qquad x_0 \ge 0$$

The number of widgets in any carton, $K$, is a random variable with the PMF

$$p_K(K_0) = \frac{\mu^{K_0}e^{-\mu}}{K_0!} \qquad K_0 = 0, 1, 2, \ldots$$

The number of cartons in a crate, $N$, is a random variable with PMF

$$p_N(N_0) = P^{N_0-1}(1 - P) \qquad N_0 = 1, 2, 3, \ldots$$

Random variables $x$, $K$, and $N$ are mutually independent.
Determine:
**a** The probability that a randomly selected crate contains exactly one widget
**b** The conditional PDF for the total weight of widgets in a carton given that the carton contains less than two widgets
**c** The $s$ transform of the PDF for the total weight of the widgets in a crate
**d** The probability that a randomly selected crate contains an odd number of widgets

**3.21** The number of customers who shop at a supermarket in a day has the PMF

$$p_k(k_0) = \frac{\lambda^{k_0}e^{-\lambda}}{k_0!} \qquad k_0 = 0, 1, 2, \ldots$$

and, independent of $k$, the number of items purchased by any customer has the PMF

$$p_l(l_0) = \frac{\mu^{l_0}e^{-\mu}}{l_0!} \qquad l_0 = 0, 1, 2, \ldots$$

Two ways the market can obtain a 10% increase in the expected value of the number of items sold are:
**a** To increase $\mu$ by 10%
**b** To increase $\lambda$ by 10%
Which of these changes would lead to the smaller variance of the total items sold per day?