Massachusetts Institute of Technology

Department of Electrical Engineering & Computer Science

6.041/6.431: Probabilistic Systems Analysis

(Spring 2006)

Solutions for Recitation 21

Markov Chains: Absorption Probabilities and Expected Time to Absorption

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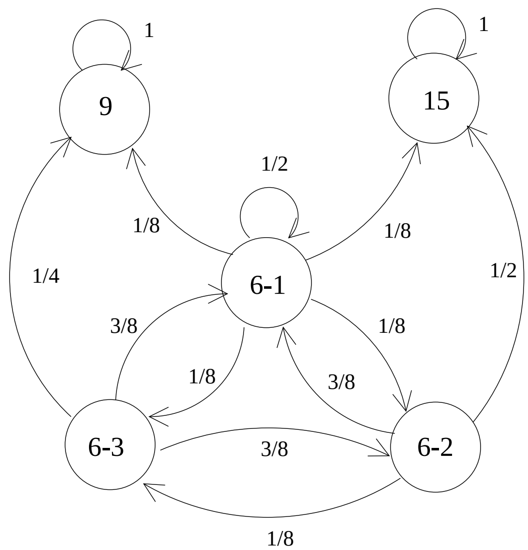
1. (a) The Markov chain is shown below.



By inspection, the states 6-1, 6-2, and 6-3 are all transient, since they each have paths leading to either state 9 or state 15, from which there is no return. Therefore she eventually leaves course 6 with probability $\boxed{1}$.

(b) This is simply the absorption probability for the recurrent class consisting of the state course-15. Let us denote the probability of being absorbed by state 15 conditioned on being in state i as a_i. Then

$$\begin{aligned}
a_{15} &= 1 \\
a_9 &= 0 \\
a_{6-1} &= \frac{1}{2}a_{6-1} + \frac{1}{8}(1) + \frac{1}{8}a_{6-2} + \frac{1}{8}(0) + \frac{1}{8}a_{6-3} \\
a_{6-2} &= \frac{1}{2}(1) + \frac{3}{8}a_{6-1} + \frac{1}{8}a_{6-3} \\
a_{6-3} &= \frac{1}{4}(0) + \frac{3}{8}a_{6-1} + \frac{3}{8}a_{6-2}
\end{aligned}$$

Solving this system of equations yields

$$a_{6-1} = \frac{105}{184} \approx 0.571$$

We will keep the other a_i's around as well - they will be useful later:

$$a_{6-2} = 0.77717$$
$$a_{6-3} = 0.50543$$

(c) This corresponds to an expected time until absorption for the transient state $6-1$. Let μ_i be the expected time until absorption conditioned on being in state i. Then

$$\begin{aligned}
\mu_{15} &= 0 \\
\mu_{9} &= 0 \\
\mu_{6-1} &= 1 + \frac{1}{2}\mu_{6-1} + \frac{1}{8}(0) + \frac{1}{8}\mu_{6-2} + \frac{1}{8}(0) + \frac{1}{8}\mu_{6-3} \\
\mu_{6-2} &= 1 + \frac{1}{2}(0) + \frac{3}{8}\mu_{6-1} + \frac{1}{8}\mu_{6-3} \\
\mu_{6-3} &= 1 + \frac{1}{4}(0) + \frac{3}{8}\mu_{6-1} + \frac{3}{8}\mu_{6-2}
\end{aligned}$$

Solving this system of equations yields

$$\mu_{6-1} = \frac{162}{46} = \frac{81}{23} \approx 3.522$$

(d) The student buys one ice cream cone every time she goes from 6-2 to 6-1 or from 6-3 to 6-1, and buys no more than 2 ice cream cones. Let us denote $v_i(j)$ as the probability that she transitions from from $6-2$ to $6-1$ or from $6-3$ to $6-1$ j times before leaving course 6, conditioned on being in state i. Then we are interested in the expected value of the random variable N, which denotes the number of cones bought before leaving course 6, and takes on the values 0, 1, or 2. So

$$\mathbf{E}[N] = (0)v_{6-1}(0) + (1)v_{6-1}(1) + (2)(1 - v_{6-1}(0) - v_{6-1}(1))$$

We use the total probability theorem, conditioning on the next day, to yield the following set of recursive equations:

$$\begin{aligned}
v_{15}(0) &= 1 \\
v_{9}(0) &= 1 \\
v_{6-1}(0) &= \frac{1}{2}v_{6-1}(0) + \frac{1}{8}v_{6-2}(0) + \frac{1}{8}v_{6-3}(0) + \frac{1}{8}(1) + \frac{1}{8}(1) \\
v_{6-2}(0) &= \frac{3}{8}(0) + \frac{1}{8}v_{6-3}(0) + \frac{1}{2}(1) \\
v_{6-3}(0) &= \frac{3}{8}(0) + \frac{3}{8}v_{6-2}(0) + \frac{1}{4}(1)
\end{aligned}$$

Solving this system of equations yields:

$$v_{6-1}(0) = \frac{46}{61} \approx 0.754$$

We still need to find v_{6-1}, and we do this by again conditioning on the second following day:

$$\begin{aligned}
v_{6-1}(1) &= \frac{1}{2}v_{6-1}(1) + \frac{1}{8}v_{6-2}(1) + \frac{1}{8}v_{6-3}(1) + \frac{1}{8}(0) + \frac{1}{8}(0) \\
v_{6-2}(1) &= \frac{3}{8}v_{6-1}(0) + \frac{1}{8}v_{6-3}(1) + \frac{1}{2}(0) \\
v_{6-3}(1) &= \frac{3}{8}v_{6-1}(0) + \frac{3}{8}v_{6-2}(1) + \frac{1}{4}(0)
\end{aligned}$$

Notice in the second and third equations that when she goes into state 6-1, this automatically sets v to 1, so we require that there be no more transitions from 6-2 to 6-1 or from 6-3 to 6-1 after the second day (that is, v=0 starting in state 6-1, whose probability we found before). Solving this system of equations yields:

$$v_{6-1}(1) = \frac{690}{3721} \approx 0.185$$

Finally, we can solve for the expected number of cones:

$$\begin{aligned}
\mathbf{E}[N] &= (0)v_{6-1}(0) + (1)v_{6-1}(0) + (2)(1 - v_{6-1}(0) - v_{6-1}(1)) \\
&= \frac{690}{3721} + 2(\frac{225}{3721}) \\
&= \frac{1140}{3721} \approx 0.306
\end{aligned}$$

(e) We want to find the expected time to absorption conditioned on the event that the student eventually ends up in state 15, which we will call A. So

$$\begin{aligned}
\mathbf{P}_{i,j|A} &= \mathbf{P}(X_{n+1} = j | X_n = i, X_\infty = 15) \\
&= \frac{\mathbf{P}(X_\infty = 15 | X_{n+1} = j)\mathbf{P}(X_{n+1} = j | X_n = i)}{\mathbf{P}(X_\infty = 15 | X_n = i)} \\
&= \frac{a_j \mathbf{P}_{i,j}}{a_i}
\end{aligned}$$

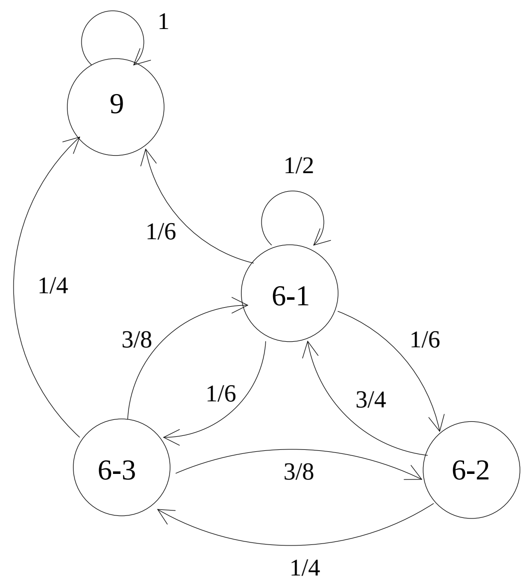
where a_k is the absorption probability of eventually ending up in state 15 conditioned on being in state k, which we found in part (b). So we may modify our chain with these new conditional probabilities and calculate the expected time to absorption on the new chain. Note that state 9 now disappears. Also, note that $\mathbf{P}_{j,j|A} = \mathbf{P}_{j,j}$, but $\mathbf{P}_{i,j|A} \neq \mathbf{P}_{i,j}$ for $i \neq j$, which means that we may not simply renormalize the transition probabilities in a uniform fashion after conditioning on this event. Let us denote the new expected time to absorption, conditioned on being in state i as $\tilde{\mu}_i$ Our system of equations now becomes

$$\begin{aligned}
\tilde{\mu}_{15} &= 0 \\
\tilde{\mu}_{6-1} &= 1 + \frac{a_{6-1}}{a_{6-1}}\frac{1}{2}\tilde{\mu}_{6-1} + 0 + \frac{a_{6-2}}{a_{6-1}}\frac{1}{8}\tilde{\mu}_{6-2} + 0 + \frac{a_{6-3}}{a_{6-1}}\frac{1}{8}\tilde{\mu}_{6-3} \\
\tilde{\mu}_{6-2} &= 1 + 0 + \frac{a_{6-1}}{a_{6-2}}\frac{3}{8}\tilde{\mu}_{6-1} + \frac{a_{6-3}}{a_{6-2}}\frac{1}{8}\tilde{\mu}_{6-3} \\
\tilde{\mu}_{6-3} &= 1 + 0 + \frac{a_{6-1}}{a_{6-3}}\frac{3}{8}\tilde{\mu}_{6-1} + \frac{a_{6-2}}{a_{6-3}}\frac{3}{8}\tilde{\mu}_{6-2}
\end{aligned}$$

Solving this system of equations yields

$$\tilde{\mu}_{6-1} = \frac{1763}{483} \approx 3.65$$

(f) The new Markov chain is shown below.



This is another expected time to absorption question on the new chain. Let us define μ_k to be the expected number of days it takes the student to go from state k to state 9 in this new Markov chain:

$$\begin{aligned}
\mu_{6-1} &= 1 + \frac{1}{2}\mu_{6-1} + \frac{1}{6}\mu_{6-2} + \frac{1}{6}\mu_{6-3} + \frac{1}{6}(0) \\
\mu_{6-2} &= 1 + \frac{3}{4}\mu_{6-1} + \frac{1}{4}\mu_{6-3} \\
\mu_{6-3} &= 1 + \frac{3}{8}\mu_{6-1} + \frac{3}{8}\mu_{6-2} + \frac{1}{4}(0)
\end{aligned}$$

Solving this system of equations yields:

$$\mu_{6-1} = \frac{86}{13} \approx 6.615$$

(g) The corresponding Markov chain is the same as the one in part (a) except $p_{9,6-1} = \frac{1}{8}, p_{9,9} = \frac{7}{8}, p_{15,6-1} = \frac{1}{8}, p_{15,15} = \frac{7}{8}$ instead of $p_{9,9} = 1, p_{15,15} = 1$.
We can consider state 6-1 as an absorbing state. Let μ_k be the expected number of transitions to be absorbed if we start at state k

$$\begin{aligned}
\mu_9 &= \frac{1}{8} + \frac{7}{8}(1 + \mu_9) \Rightarrow \mu_9 = 8 \\
\mu_{15} &= \frac{1}{8} + \frac{7}{8}(1 + \mu_{15}) \Rightarrow \mu_{15} = 8 \\
\mu_{6-3} &= \frac{3}{8} + \frac{3}{8}(1 + \mu_{6-2}) + \frac{1}{4}(1 + \mu_9) \\
\mu_{6-2} &= \frac{3}{8} + \frac{1}{8}(1 + \mu_{6-3}) + \frac{1}{2}(1 + \mu_{15}) \\
&\Rightarrow \mu_{6-2} = \frac{344}{61}, \mu_{6-3} = \frac{312}{61}
\end{aligned}$$

Let R be the number of days until she is 6-1 again. We find $E[R]$ by using the total expectation theorem, conditioned on what happens on the first transition.

$$
\begin{aligned}
\mathbf{E}[R] &= \mathbf{E}[\mathbf{E}[R|X_2]] \\
&= \frac{1}{2}(1) + \frac{1}{8}(1+\mu_9) + \frac{1}{8}(1+\mu_{15}) + \frac{1}{8}(1+\mu_{6-2}) + \frac{1}{8}(1+\mu_{6-3}) \\
&= \frac{265}{61}
\end{aligned}
$$

Notice that this chain consists of a single recurrent aperiodic class. Another approach to solving this problem uses the steady state probabilites of this chain, which are $\pi_{6-1} = \frac{61}{265}, \pi_{6-2} = \frac{11}{265}, \pi_{6-3} = \frac{9}{265}, \pi_9 = \frac{79}{265}, \pi_{15} = \frac{105}{265}$. The expected frequency of visits to 6-1 is π_{6-1}, so the expected number of days between visits to 6-1 is $\frac{1}{\pi_{6-1}}$. Since she is currently 6-1, the expected number of days until she is 6-1 again is $\frac{1}{\pi_{6-1}} = \frac{265}{61}$.