Massachusetts Institute of Technology
Department of Electrical Engineering & Computer Science
6.041/6.431: Probabilistic Systems Analysis
(Spring 2006)

Solutions to Final Exam: Spring 2006

Problem 1

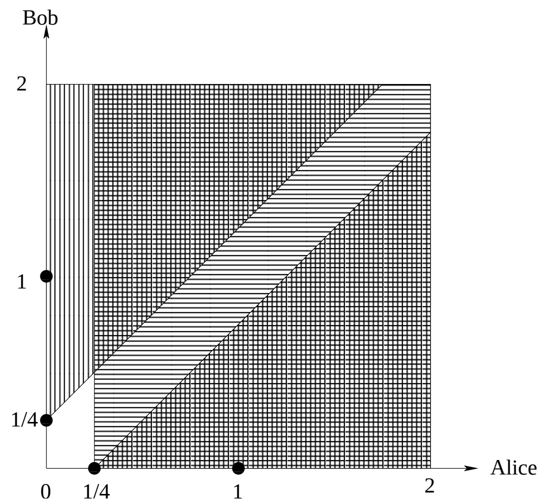
(42 points, Each question is equally weighted at 3.5 points each)

Multiple Choice Questions: There is only one correct answer for each question listed below, please clearly indicate the correct answer.

(1) **D**

Alice's and Bob's choices of number can be illustrated in the following figure. Event A (the absolute difference of the two numbers is greater than $\frac{1}{4}$) corresponds to the area shaded by the vertical line. Event B (Alice's number is greater than $\frac{1}{4}$) corresponds to the area shaded by the horizontal line. There fore, event A∩B corresponds to the double shaded area. Since both choices is uniformly distributed between 0 and 2, we have that

$$\begin{aligned}\mathbf{P}(A \cap B) &= \frac{\text{Double shaded area}}{\text{Total shaded area}} \\ &= \frac{\frac{1}{2} \times \left(\frac{3}{2}\right)^2 + \frac{1}{2} \times \left(\frac{7}{4}\right)^2}{4} \\ &= \frac{85}{128}.\end{aligned}$$



(2) **B**

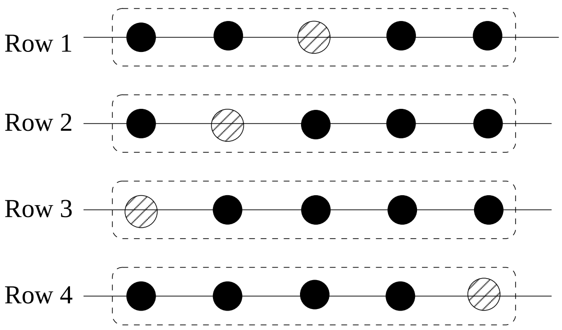
Let B_1 be the first ball and B_2 be the second ball (you may do this by drawing two balls with both hands, then first look at the ball in left hand and then look at the ball in right hand).

$$\begin{aligned}
& \mathbf{P}(\ B_1 \text{ and } B_2 \text{ are of different color }) \\
= \ & \mathbf{P}(\ B_1 \text{ is red })\mathbf{P}(\ B_2 \text{ is white } | \ B_1 \text{ is red }) + \mathbf{P}(\ B_1 \text{ is white })\mathbf{P}(\ B_2 \text{ is red } | \ B_1 \text{ is white }) \\
= \ & \frac{m}{m+n} \times \frac{n}{m+n-1} + \frac{n}{m+n} \times \frac{m}{m+n-1} \\
= \ & \frac{2\,m\,n}{(m+n)(m+n-1)}
\end{aligned}$$

(3) **C**

Without any constraint, there are totally $20 \times 19 \times 18 \times 17$ possible ways to get 4 pebbles out of 20. Now we consider the case under the constraint that the 4 picked pebbles are on different rows. As illustrated by the following figure, there are 20 possible positions for the first pebble, 15 for the second, 10 for the third and finally 5 for the last pebble. Therefore, there are a total of $20 \times 15 \times 10 \times 5$ possible ways. With the uniform discrete probability law, the probability of picking 4 pebbles from different rows is

$$\frac{20 \times 15 \times 10 \times 5}{\frac{20!}{16!}} = \frac{5^4 \cdot 4! \cdot 16!}{20!}.$$



(4) **A**

Let X_A and X_B denote the the life time of bulb A and bulb B, respectively. Clearly, X_A has exponential distribution with parameter $\lambda_A = \frac{1}{4}$ and X_B has exponential distribution with parameter $\lambda_B = \frac{1}{6}$. Let E be the event that we have selected bulb A. We use X to denote the lifetime of the selected bulb. Using Bayes' rule, we have that

$$\begin{aligned}
\mathbf{P}(E \,|\, X \geq \frac{1}{2}) &= \frac{\mathbf{P}(E) \cdot \mathbf{P}(X \geq \frac{1}{2} \,|\, E)}{\mathbf{P}(X \geq \frac{1}{2})} \\
&= \frac{\frac{1}{2} e^{-\lambda_A \cdot \frac{1}{2}}}{\frac{1}{2} e^{-\lambda_A \cdot \frac{1}{2}} + \frac{1}{2} e^{-\lambda_B \cdot \frac{1}{2}}}
\end{aligned}$$

$$= \frac{1}{1+e^{\frac{1}{24}}}.$$

(5) **C**

We define the following events:

D	A person has the disease,
D^c	A person doesn't have the disease,
T	A person is tested positive,
T^c	A person is tested negative.

We know that $\mathbf{P}(D) = 0.001$, $\mathbf{P}(T \mid D) = 0.95$, $\mathbf{P}(T^c \mid D^c) = 0.95$. Therefore, using the Bayes' rule, we have

$$\begin{aligned} \mathbf{P}(D \mid T) &= \frac{\mathbf{P}(D) \cdot \mathbf{P}(T \mid D)}{\mathbf{P}(T)} \\ &= \frac{0.001 \cdot 0.95}{0.001 \cdot 0.95 + 0.999 \cdot 0.05} \\ &= 0.0187 . \end{aligned}$$

(6) **D**

$$\begin{aligned} F_Y(y) &= \mathbf{P}(Y \le y) \\ &= \mathbf{P}(X \le \frac{y}{2} + 2) \\ &= F_X(\frac{y}{2} + 2) . \end{aligned}$$

Take derivative on both side of the equation, we have that $f_Y(y) = \frac{1}{2} f_X(\frac{y}{2} + 2)$. Therefore, $f_Y(0) = \frac{1}{2} f_X(2) = e^{-4}$.

(7) **B**

Clearly, X is exponentially distributed with parameter $\lambda = 3$. Therefore, its transform is $M_X(s) = \frac{3}{3-s}$. The transform of Y can be easily calculated from its PMF as $M_Y(s) = \frac{1}{2}e^s + \frac{1}{2}$. Since X and Y are independent, transform of the X+Y is just the product of the transform of X and the transform of Y. Thus,

$$\begin{aligned} M_Z(s)\big|_{s=1} &= M_X(s)\big|_{s=1} \cdot M_Y(s)\big|_{s=1} \\ &= \frac{3}{2} \cdot \left(\frac{1}{2} + \frac{1}{2}e\right) \\ &= \frac{3}{4}(1+e) . \end{aligned}$$

(8) **C**

Let P be the random variable for the value drawn according to the uniform distribution in interval $[0,1]$ and let X be the number of successes in k trials. Given $P = p$, X is a binomial random variable:

$$p_{X|P}(x|p) = \begin{cases} \binom{k}{x} p^x (1-p)^{k-x} & x = 0, 1, ..., k \\ 0 & \text{otherwise.} \end{cases}$$

From the properties of a binomial r.v. we know that $\mathbf{E}[X|P = p] = kp$, and $\text{var}(X|P = p) = kp(1-p)$.

Now let's find $\text{var}(X)$ using the law of total variance:

$$\begin{aligned} \text{var}(X) &= \mathbf{E}[\text{var}(X|P)] + \text{var}(\mathbf{E}[X|P]) \\ &= \mathbf{E}[kP(1-P)] + \text{var}(kP) \\ &= k(\mathbf{E}[P] - \mathbf{E}[P^2]) + k^2\text{var}(P) \\ &= k[\frac{1}{2} - (\frac{1}{12} + 14)] + k^2\frac{1}{12} \\ &= \frac{k}{6} + \frac{k^2}{12} \end{aligned}$$

Therefore the variance is: $\text{var}(X) = \mathbf{E}[X^2] - \mathbf{E}[X]^2 = \frac{k}{6} + \frac{k^2}{12}$.

(9) **B**

Let X be the car speed and let Y be the radar's measurement. Then, the joint PDF of X and Y is uniform in the range of pairs (x, y) such that $x \in [60, 75]$ and $x \leq y \leq x + 5$. Therefore, the least square estimator of X given $Y = y$ is

$$\mathbf{E}[\, X \,|\, Y = y \,] \quad = \quad \begin{cases} \frac{1}{2}y + 30 & 60 \leq y \leq 65\,, \\ y - 2.5 & \text{if } 65 \leq y \leq 75\,, \\ \frac{1}{2}y + 35 & \text{if } 75 \leq y \leq 80\,. \end{cases} \tag{1}$$

Thus, the LSE of X when $Y = 76$ is $\mathbf{E}[\, X \,|\, Y = 76\,] = \frac{1}{2} \times 76 + 35 = 73$.

(10) **A**

The mosquito's arrivals form a Bernoulli process with parameter p. Since the expected time till the first bite is 10 seconds, the parameter p is equal to $\frac{1}{10}$. The time when you die is the time when the second mosquito arrives, which is the second arrival time of the mosquito arrival process, denoted as T_2. T_2's PMF is the order 2 Pascal. Thus, the probability that you die at time 10 seconds is

$$\begin{aligned} \mathbf{P}_{T_2}(10) &= \mathbf{P}_{T_2}(t)|_{t=10} \\ &= \left(\begin{array}{c} t-1 \\ 2-1 \end{array} \right) p^2(1-p)^{t-2} \Big|_{t=10} \\ &= \left(\begin{array}{c} 10-1 \\ 2-1 \end{array} \right) \frac{1}{10}^2 (1 - \frac{1}{10})^{10-2} \\ &= \frac{9^9}{10^{10}} \end{aligned}$$

(11) **B**

Let X_i be the indicator random variable for the i^{th} person's vote as follows:

$$X_i = \begin{cases} 1 & \text{if the } i^{\text{th}} \text{ person will vote for Bush}\,, \\ 0 & \text{if the } i^{\text{th}} \text{ person will not vote for Bush}\,. \end{cases}$$

Since the n people vote independently, the X_is are mutually independent, with $\mathbf{E}[X_i] = p$ and $\text{var}(X_i) = p(1-p)$. Since $M_n = \frac{S_n}{n} = \frac{X-1+\ldots+X_n}{n}$, we have that $\mathbf{E}[M_n] = p$ and $\text{var}(M_n) = \frac{\text{var}(X_i)}{n} = \frac{p(1-p)}{n}$. Using Chebyshev inequality,

$$\begin{aligned} \mathbf{P}(|M_n - p| \geq 0.01) &\leq \frac{\text{var}(M_n)}{(0.01)^2} \\ &= \frac{\frac{1}{n}p(1-p)}{(0.01)^2} \\ &\leq \frac{1}{n} \cdot \frac{1}{4} \times 10^4\,. \end{aligned}$$

In the last step we use the fact that $p(1-p) \leq \frac{1}{4}$ for $p \in [0,1]$. Thus, if $\frac{1}{n} \cdot \frac{1}{4}^4 \leq 0.01$ or equivalently $n \geq 250{,}000$, then $\mathbf{P}(|M_n - p| \geq 0.01) \leq 0.01$ is satisfied.

(12) **D**

Let X_i be the indicator random variable for the i^{th} day being rainy as follows:

$$X_i = \begin{cases} 1 & \text{if it rains on the } i^{\text{th}} \text{ day}\,, \\ 0 & \text{if it does not rain on the } i^{\text{th}} \text{ day}\,. \end{cases}$$

We have that $\mu = \mathbf{E}[X_i] = 0.1$ and $\sigma^2 = \text{var}(X_i) = 0.1 \cdot (1 - 0.1) = 0.09$. Then the number of rainy days in a year is $S_{365} = X_1 + \ldots + X_{365}$, which is distributed as a binomial$(n = 365, p = 0.1)$. Using the central limit theorem, we have that

$$\begin{aligned} \mathbf{P}(S_{365} \geq 100) &= \mathbf{P}\left(\frac{S_365 - 365 \cdot \mu}{\sigma\sqrt{365}} \geq \frac{100 - 365 \cdot \mu}{\sigma\sqrt{365}}\right) \\ &= \mathbf{P}\left(\frac{S_365 - 365 \cdot \mu}{\sigma\sqrt{365}} \geq \frac{100 - 365 \cdot 0.1}{0.3\sqrt{365}}\right) \\ &\approx 1 - \Phi\left(\frac{635}{3\sqrt{365}}\right)\,. \end{aligned} \tag{2}$$

Problem 2

(30 points, Each question is equally weighted at 6 points each)

The deer process is a Poisson process of arrival rate 8 ($\lambda_D = 8$). The elephant process is a Poisson process of arrival rate 2 ($\lambda_E = 2$).

(a) $\boxed{30}$

Since deer and elephants are the only animals that visit the river, then the animal process is the merged process of the deer process and the elephant process. Since the deer process and the elephant process are independent Poisson processes with $\lambda_D = 8$ and $\lambda_E = 2$ respectively, the animal process is also a poisson process with $\lambda_A = \lambda_D + \lambda_E = 10$. Therefore, the number of animals arriving the three hours is distributed according to a Poisson distribution with parameter $\lambda_A \cdot 3 = 30$. Therefore,

$$\begin{aligned} \mathbf{E}[\text{ number of animals arriving in three hours }] &= \lambda_A \cdot 3 = 30\,, \\ \text{var}(\text{ number of animals arriving in three hours }) &= \lambda_A \cdot 3 = 30\,. \end{aligned}$$

(b) $\boxed{\sum_{k=3}^{11} \binom{k-1}{2} \left(\frac{1}{5}\right)^3 \left(\frac{4}{5}\right)^{k-3}}$

Let's now focus on the animal arrivals themselves and ignore the inter-arrival times. Each animal arrival can be either a deer arrival or a elephant arrival. Let E_k denote the event that the k^{th} animal arrival is an elephant. It is explained in the book (Example 5.14, Page 295) that

$$\mathbf{P}(E_k) = \frac{\lambda_E}{\lambda_D + \lambda_E} = \frac{1}{5}\,,$$

and the events $E_1, E_2, \ldots$ are independent. Regarding the arrival of elephant as "success", it forms a Bernoulli process, with parameter $p = \frac{1}{5}$. Clearly, observing the 3^{rd} elephant before the 9^{th} deer is equivalent to that the 3^{rd} elephant arrives before time 12 in the above Bernoulli process. Let T_3 denote the 3^{rd} elephant arrival time in the Bernoulli process.

$$\begin{aligned} \mathbf{P}[T_3 < 12] &= \sum_{k=3}^{11} \mathbf{P}[T_3 = k] \\ &= \sum_{k=3}^{11} \binom{k-1}{2} p^3 (1-p)^{k-3} \\ &= \sum_{k=3}^{11} \binom{k-1}{2} \left(\frac{1}{5}\right)^3 \left(\frac{4}{5}\right)^{k-3}\,. \end{aligned}$$

(c) $\boxed{1 - e^{-2}}$

Since the deer arrival process is independent of the elephant arrival process, the event that "several deer arrive by the end of 3 hours" is independent of any event defined on the elephant arrival process. Let T_1 denote the first arrival time. Then the desired probability can be expressed as

$$\begin{aligned} & \mathbf{P}[\text{ an elephant arrival in the next hour} \mid \text{no elephant arrival by the end of 3 hours }] \\ = \; & \mathbf{P}[\,T_1 \leq 4 \,|\, T_1 > 3\,] \\ = \; & 1 - \mathbf{P}[\,T_1 > 4 \,|\, T_1 > 3\,] \\ = \; & 1 - \mathbf{P}[\,T_1 > 1\,] \qquad (3) \\ = \; & 1 - e^{-2}\,. \end{aligned}$$

Step (3) follows from the memoryless property of the Poisson process.

(d) **4**

As in part (b), regarding the arrival of elephant as "success", it forms a Bernoulli process, with parameter $p = \frac{1}{5}$. Owing to the memoryless property, given that no "success" in the first 53 trials (have seen 53 deer but no elephant), the number of trials, X_1, up to and including the first "success" is still a geometrical random variable with parameter $p = \frac{1}{5}$. Since the number of "failures" (deers) before the first "success" (elephant) is $X_1 - 1$, then

$$\mathbf{E}[\text{ number of more deers before the first elephant }] = \mathbf{E}[\,X_1 - 1\,] = \frac{1}{p} - 1 = 4\,.$$

(e) $\mathbf{\frac{21}{40}}$

Let T_D (resp. T_E) denote the first arrival time of the deer process (resp. elephant process). The time S until I clicked a picture of both a deer and a elephant is equal to $\max\{T_D, T_E\}$. We express S into two parts,

$$S = \max\{T_D, T_E\} = \min\{T_D, T_E\} + (\max\{T_D, T_E\} - \min\{T_D, T_E\}) = S_1 + S_2\,,$$

where $S_1 = \min\{T_D, T_E\}$ is the first arrival time of an animal and $S_2 = \max\{T_D, T_E\} - \min\{T_D, T_E\}$ is the additional time until both animals register one arrival. Since the animal arrival process is Poisson with rate 10,

$$\mathbf{E}[S_1] = \frac{1}{\lambda_D + \lambda_E}\,. \qquad (4)$$

Concerning S_2, there are two cases.

(1) The first arrival is a deer, which happens with probability $\frac{\lambda_D}{\lambda_D + \lambda_E}$. Then we wait for an elephant arrival, which takes $\frac{1}{\lambda_E}$ time on average.

(2) The first arrival is an elephant, which happens with probability $\frac{\lambda_E}{\lambda_D + \lambda_E}$. Then we wait for a deer arrival, which takes $\frac{1}{\lambda_D}$ time on average.

Putting everything together, we have that

$$\begin{aligned}\mathbf{E}[S] &= \frac{1}{\lambda_D+\lambda_E} + \frac{\lambda_D}{\lambda_D+\lambda_E}\cdot\frac{1}{\lambda_E} + \frac{\lambda_E}{\lambda_D+\lambda_E}\cdot\frac{1}{\lambda_D} \\ &= \frac{1}{10} + \frac{4}{5}\cdot\frac{1}{2} + \frac{1}{5}\cdot\frac{1}{8} \\ &= \frac{21}{40}.\end{aligned}$$

(f) $f_X(x) = \begin{cases} \frac{(0.9\cdot 2)^3 x^2 e^{-1.8x}}{2} & x > 0\,, \\ 0 & \text{otherwise}\,. \end{cases}$

X denotes the time until the third successful picture of an elephant. Since the deer arrivals are independent of the elephant arrivals and X is only related to elephant arrivals, we can focus on the elephant arrival process. We split the elephant arrival process into two processes, one composed of the successful elephant pictures (We call this process the "successful elephant picture arrival process") and the other one composed of unsuccessful elephant pictures (We call this process the "unsuccessful elephant picture arrival process"). Since the quality of each picture is independent of everything else, the successful elephant picture arrival process is a Poisson process with parameter $\lambda_{SE} = 0.9\cdot\lambda_E$. Therefore, X is the 3^{rd} arrival time of the successful elephant picture arrival process. Thus X has the following Erlang distribution

$$f_X(x) = \begin{cases} \frac{(0.9\cdot 2)^3 x^2 e^{-1.8x}}{2} & x > 0\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

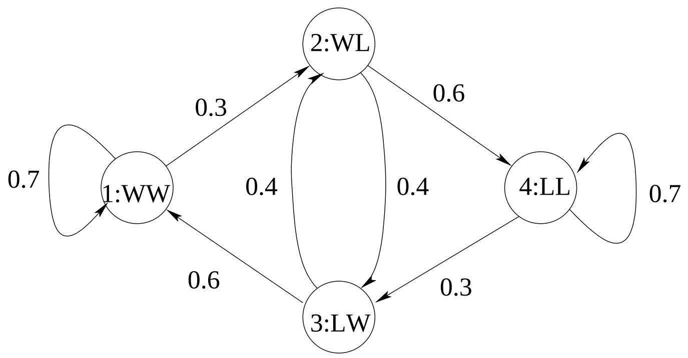
Problem 3

(28 points, Each question is equally weighted at 4 points each)

(a) The performance of the MIT football team is described by a Markov chain, where the state is taken to be

$$(\textit{result of the game before the last game}, \quad \textit{result of the last game})\,.$$

There are four possible states: $\{WW, WL, LW, LL\}$. The corresponding transition probability graph is We use throughout this problem the following labels for the four states.



$$WW = 1\,, \qquad WL = 2\,, \qquad LW = 3\,, \qquad LL = 4\,.$$

Then we have the probability transition matrix as follows

$$\mathbf{P} = [\,p_{ij}\,]_{4\times 4} = \begin{pmatrix} 0.7 & 0.3 & 0 & 0 \\ 0 & 0 & 0.4 & 0.6 \\ 0.6 & 0.4 & 0 & 0 \\ 0 & 0 & 0.3 & 0.7 \end{pmatrix},$$

where p_{ij} is the transition probability from state i to state j.

(b) $\boxed{\mathbf{0.6}}$

When the winning streak of the team gets interrupted, the chain is in state 2. The probability of losing the next game is 0.6.

An alternative computation-intensive method: Denote the starting time to be $n = 1$ and let X_n denote the result of the n^{th} game. Since the team starts when they has won their previous two games, we set $X_0 = X_{-1} = W$. Then,

$$\begin{aligned} & \mathbf{P}[\text{ the first future loss followed by another loss } \mid X_0 = X_{-1} = W\,] \\ = & \sum_{n=1}^{+\infty} \mathbf{P}[\text{ the first future loss occurs at time } n \text{ and another loss at time } n+1 \mid X_0 = X_{-1} = W\,] \end{aligned}$$

$$\begin{aligned}
&= \sum_{n=1}^{+\infty} \mathbf{P}[\,X_{n+1} = X_n = L, X_{n-1} = \ldots = X_1 = W \,|\, X_0 = X_{-1} = W\,] \\
&= \sum_{n=1}^{+\infty} p_{11}^{n-1} \cdot p_{12} \cdot p_{24} \\
&= \sum_{n=1}^{+\infty} 0.7^{n-1} \cdot 0.3 \cdot 0.6 \\
&= 0.6\,.
\end{aligned}$$

(c) $\boxed{p_X(k) = \begin{cases} 0.7^k \cdot 0.3 & k = 0, 1, 2, \ldots \\ 0 & \text{o.w.} \end{cases}}$

Clearly, X, the number of games played before the first loss, takes value in $\{0, 1, \ldots\}$. Using the same notations as in (b),the PMF of X can be calculated as follows.

$$\begin{aligned}
p_X(k) &= \mathbf{P}[\,X_{k+1} = L, X_k = \ldots = X_1 = W \,|\, X_0 = X_{-1} = W\,] \\
&= p_{11}^k \cdot p_{12}\,.
\end{aligned}$$

Therefore, X's PMF is

$$p_X(k) = \begin{cases} 0.7^k \cdot 0.3 & k = 0, 1, 2, \ldots \\ 0 & \text{o.w.} \end{cases}$$

(d) $\boxed{\pi_1 = \pi_4 = \frac{1}{3}, \pi_2 = \pi_3 = \frac{1}{6}}$

The Markov chain consists of a single aperiodic recurrent class, so the steady-state distribution exists, denoted by $\vec{\pi} = (\pi_1, \pi_2, \pi_3, \pi_4)$. Solving the equilibrium system $\vec{\pi}\mathbf{P} = \vec{\pi}$ together with $\sum_{i=1}^{4} \pi_i = 1$, we get the desired result.

(e) $\boxed{\frac{1}{2}}$

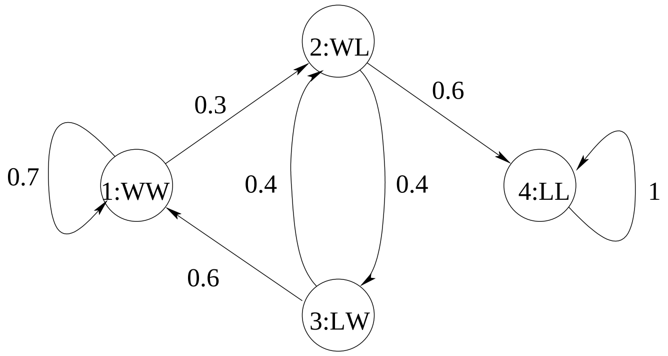
Recall that X_n denotes the outcome of the n^{th} game. The desired probability is

$$\begin{aligned}
&\mathbf{P}[\,X_{1000} = W \,|\, X_{1000} = X_{1001}\,] \\
&= \frac{\mathbf{P}[\,X_{1000} = X_{1001} = W\,]}{\mathbf{P}[\,X_{1000} = X_{1001}\,]} \\
&= \frac{\mathbf{P}[\,X_{1000} = X_{1001} = W\,]}{\mathbf{P}[\,X_{1000} = X_{1001} = W\,] + \mathbf{P}[\,X_{1000} = X_{1001} = L\,]} \\
&\approx \frac{\pi_1}{\pi_1 + \pi_4}\,, \qquad (5)
\end{aligned}$$

which evaluates to be $\frac{1}{2}$.

(f) $$\begin{cases} t_1 &= 1 + 0.7 \cdot t_1 + 0.3 \cdot t_2 \\ t_2 &= 1 + 0.4 \cdot t_3 \\ t_3 &= 1 + 0.6 \cdot t_1 + 0.4 \cdot t_2 \\ t_4 &= 0 \end{cases}.$$

T is the first passage time from state 1 to state 4. Let $\mu_1, \mu_2, \mu_3, \mu_4$ be the average first passage time from each state to state 4. We have that $\mathbf{E}[T] = \mu_1$ Since we are concerned with the first passage time to state 4, the Markov chain's behavior from state 4 is irrelevant. Therefore, we focus on the modified Markov chain graph where state 4 is converted to an absorbing state. The average first



passage time to state 4 in the original Markov chain is equal to the average absorption time to state 4 in the modified Markov chain. The required linear equation system is

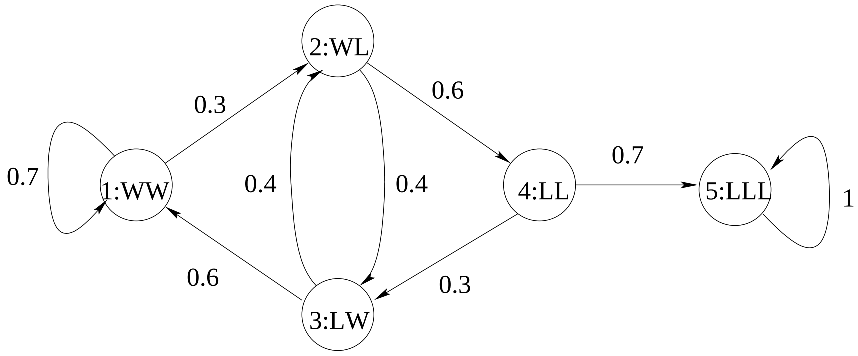
$$\begin{cases} t_1 &= 1 + 0.7 \cdot t_1 + 0.3 \cdot t_2 \\ t_2 &= 1 + 0.4 \cdot t_3 \\ t_3 &= 1 + 0.6 \cdot t_1 + 0.4 \cdot t_2 \\ t_4 &= 0 \end{cases}.$$

(Though not asked in the problem, we get the solution that $\mu_1 = 7, \mu_2 = \frac{11}{3}, \mu_3 = \frac{20}{3}$).

(g) $$\begin{cases} t_1 &= 1 + 0.7t_1 + 0.3t_2 \\ t_2 &= 1 + 0.4t_3 + 0.6t_4 \\ t_3 &= 1 + 0.6t_1 + 0.4t_2 \\ t_4 &= 1 + 0.3t_3 \\ t_5 &= 0 \end{cases},$$

Three losses in a row can only be reached from state LL. Following this observation, we create a fifth state LLL (labeled 5) and construct the following Markov transition graph. The probability transition matrix of this Markov chain is

$$\mathbf{P} = [p_{ij}]_{4\times 4} = \begin{pmatrix} 0.7 & 0.3 & 0 & 0 & 0 \\ 0 & 0 & 0.4 & 0.6 & 0 \\ 0.6 & 0.4 & 0 & 0 & 0 \\ 0 & 0 & 0.3 & 0 & 0.7 \\ 0 & 0 & 0.3 & 0 & 1 \end{pmatrix},$$



The number of games played by the MIT football team corresponds to the absorption time from state 1 to state 5. Denote the average absorption time to state 5 as t_1, t_2, t_3, t_4, t_5. The linear equation system is as follows.

$$\begin{cases} t_1 &= 1 + 0.7t_1 + 0.3t_2 \\ t_2 &= 1 + 0.4t_3 + 0.6t_4 \\ t_3 &= 1 + 0.6t_1 + 0.4t_2 \\ t_4 &= 1 + 0.3t_3 \\ t_5 &= 0 \end{cases} .$$