Massachusetts Institute of Technology
Department of Electrical Engineering & Computer Science
6.041/6.431: Probabilistic Systems Analysis
(Spring 2006)

Problem Set 10
Topics: Poisson, Markov chains
Due: May 10th, 2006

1. All ships travel at the same speed through a wide canal. Eastbound ship arrivals at the canal are a Poisson process with an average arrival rate λ_E ships per day. Westbound ships arrive as an independent Poisson process with average arrival rate λ_W per day. An indicator at a point in the canal is always pointing in the direction of travel of the most recent ship to pass it. Each ship takes t days to traverse the length of the canal.

   (a) Given that the pointer is pointing west:

      i. What is the probability that the next ship to pass it will be westbound?

      ii. What is the PDF for the remaining time until the pointer changes direction?

   (b) What is the probability that an eastbound ship will see no westbound ships during its eastward journey through the canal?

   (c) We begin observing at an arbitrary time. Let V be the time we have to continue observing until we see the seventh eastbound ship. Determine the PDF for V.

2. (a) Shuttles depart from New York to Boston every hour on the hour. Passengers arrive according to a Poisson process of rate λ per hour. Find the expected number of passengers on a shuttle. (Ignore issues of limited seating.)

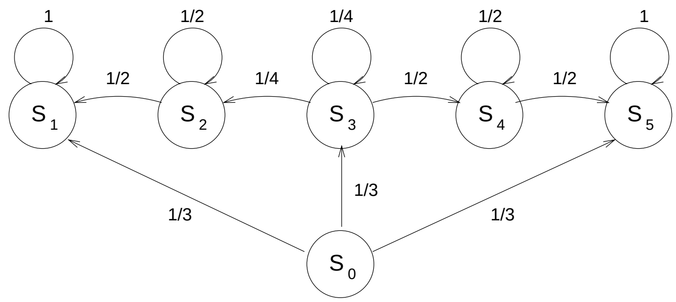
   (b) Now suppose that the shuttles are no longer operating on a deterministic schedule, but rather their interdeparture times are independent and exponentially distributed with rate μ per hour. Find the PMF for the number of shuttles arriving in one hour.

   (c) Let us define an "event" in the airport to be either the arrival of a passenger, or the departure of a plane. With the same assumptions as in (b) above, find the expected number of "events" that occur in one hour.

   (d) If a passenger arrives at the gate, and sees 2λ people waiting, find his/her expected time to wait until the next shuttle.

   (e) Find the PMF for the number of people on a shuttle.

3. Consider the following Markov chain:



Given that the above process is in state S_0 just before the first trial, determine by inspection the probability that:

(a) The process enters S_2 for the first time as the result of the kth trial.

(b) The process never enters S_4.

(c) The process enters S_2 and then leaves S_2 on the next trial.

(d) The process enters S_1 for the first time on the third trial.

(e) The process is in state S_3 immediately after the nth trial.

4. (a) Identify the transient, recurrent, and periodic states of the discrete state discrete-transition Markov process described by

$$[p_{ij}] = \begin{bmatrix} 0.5 & 0 & 0 & 0 & 0.5 & 0 & 0 \\ 0.3 & 0.4 & 0 & 0 & 0.2 & 0.1 & 0 \\ 0 & 0 & 0.6 & 0.2 & 0 & 0.2 & 0 \\ 0 & 0 & 0 & 0.5 & 0 & 0 & 0.5 \\ 0.3 & 0.4 & 0 & 0 & 0.3 & 0 & 0 \\ 0 & 0 & 0.4 & 0.6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.6 & 0 & 0 & 0.4 \end{bmatrix}$$

(b) How many classes are formed by the recurrent states of this process?

(c) Evaluate $\lim_{n\to\infty} p_{41}(n)$ and $\lim_{n\to\infty} p_{66}(n)$.

5. Out of the d doors of my house, suppose that in the beginning $k > 0$ are unlocked and $d - k$ are locked. Every day, I use exactly one door, and I am equally likely to pick any of the d doors. At the end of the day, I leave the door I used that day locked.

(a) Show that the number of unlocked doors at the end of day n, L_n, evolves as the state in a Markov process for $n \geq 1$. Write down the transition probabilities p_{ij}.

(b) List transient and recurrent states.

(c) Is there an absorbing state? How does $r_{ij}(n)$ behave as $n \to \infty$?

(d) Now, suppose that each day, if the door I pick in the morning is locked, I will leave it unlocked at the end of the day, and if it is initially unlocked, I will leave it locked. Repeat parts (a)-(c) for this strategy.

(e) My third strategy is to alternate between leaving the door I use locked one day and unlocked the next day (regardless of the initial condition of the door.) In this case, does the number of unlocked doors evolve as a Markov chain, why/why not?

G1†. Consider a Markov chain $\{X_k\}$ on the state space $\{1, \ldots, n\}$, and suppose that whenever the state is i, a reward $g(i)$ is obtained. Let R_k be the total reward obtained over the time interval $\{0, 1, \ldots, k\}$, that is, $R_k = g(X_0) + g(X_1) + \cdots + g(X_k)$. For every state i, let

$$m_k(i) = E[R_k \mid X_0 = i],$$

and

$$v_k(i) = \text{var}(R_k \mid X_0 = i)$$

respectively be the conditional mean and conditional variance of R_k, conditioned on the initial state being i.

(a) Find a recursion that, given the values of $m_k(1), \ldots, m_k(n)$, allows the computation of $m_{k+1}(1), \ldots, m_{k+1}(n)$.

(b) Find a recursion that, given the values of $m_k(1), \ldots, m_k(n)$ and $v_k(1), \ldots, v_k(n)$, allows the computation of $v_{k+1}(1), \ldots, v_{k+1}(n)$. *Hint*: Use the law of total variance.

G2†. The parking garage at MIT has installed a card operated gate, which, unfortunately, is vulnerable to absent-minded faculty and staff. In particular, in each day a car crashes the gate with probability p, in which case a new gate must be installed. Also a gate that has survived for m days must be replaced as a matter of periodic maintenance. What is the steady-state expected frequency of gate replacements?