Problem Set 4
Due: March 8, 2006

1. Professor May B. Right often has her science facts wrong, and answers each of her students' questions incorrectly with probability 1/4, independently of other questions. In each lecture Professor Right is asked either 1 or 2 questions with equal probability.

   (a) What is the probability that Professor Right gives wrong answers to all the questions she gets in a given lecture?

   (b) Given that Professor Right gave wrong answers to all the questions she was asked in a given lecture, what is the probability that she got two questions?

   (c) Let X and Y be the number of questions asked and the number of questions answered correctly in a lecture, respectively. What are the mean and variance of X and the mean and the variance of Y?

   (d) Give a neatly labeled sketch of the joint PMF $p_{X,Y}(x, y)$.

   (e) Let $Z = X + 2Y$. What are the expectation and variance of Z?

   For the remaining parts of this problem, assume that Professor Right has 20 lectures each semester and each lecture is independent of any other lecture.

   (f) The university where Professor Right works has a peculiar compensation plan. For each lecture, she gets paid a base salary of $\$1,000$ plus $\$40$ for each question she answers and an additional $\$80$ for each of these she answers correctly. In terms of random variable Z, she gets paid $\$1000 + \$40Z$ per lecture. What are the expected value and variance of her *semesterly* salary?

   (g) Determined to improve her reputation, Professor Right decides to teach an additional 20-lecture class in her specialty (math), where she answers questions incorrectly with probability 1/10 rather than 1/4. What is the expected number of questions that she will answer wrong in a randomly chosen lecture (math or science).

2. The joint PMF of discrete random variables X and Y is given by

$$p_{X,Y}(x, y) = \begin{cases} Cx^2\sqrt{y}, & \text{if } x = -5, -4, \ldots, 4, 5 \text{ and } y = 0, 1, \ldots, 10; \\ 0 & \text{otherwise.} \end{cases}$$

   Here, C is some constant. What is $\mathbf{E}[XY^3]$?
   Hint: This question admits a short answer/explanation. Don't spend time doing calculations.

3. Joe Lucky plays the lottery on any given week with probability p, independently of whether he played on any other week. Each time he plays, he has a probability q of winning, again independently of everything else. During a fixed time period of n weeks, let X be the number of weeks that he played the lottery and Y the number of weeks that he won.

   (a) What is the probability that he played the lottery any particular week, given that he did not win anything that week?

   (b) Find the conditional PMF $p_{Y|X}(y \mid x)$.

(c) Find the joint PMF $p_{X,Y}(x, y)$.

(d) Find the marginal PMF $p_Y(y)$. *Hint:* One possibility is to start with the answer to part (c), but the algebra can be messy. But if you think intuitively about the procedure that generates Y, you may be able to guess the answer.

(e) Find the conditional PMF $p_{X|Y}(x \mid y)$. Do this algebraically using previous answers.

(f) Rederive the answer to part (e) by thinking as follows: For each one of the $n - Y$ weeks that he did not win, the answer to part (a) should tell you something.

In all parts of this problem, make sure to indicate the range of values for which your PMF formula applies.

4. Let X and Y be independent random variables that take values in the set $\{1, 2, 3\}$. Let $V = 2X + 2Y$ and $W = X - Y$.

(a) Assume that $\mathbf{P}(\{X = k\})$ and $\mathbf{P}(\{Y = k\})$ are positive for any $k \in \{1, 2, 3\}$. Can V and W be independent? Explain. (No calculations needed.)

For the remaining parts of this problem, assume that X and Y are uniformly distributed on $\{1, 2, 3\}$.

(b) Find and plot $p_V(v)$. Also, determine $\mathbf{E}[V]$ and $\text{var}(V)$.

(c) Find and show in a diagram $p_{V,W}(v, w)$.

(d) Find $\mathbf{E}[V \mid W > 0]$.

(e) Find the conditional variance of W given the event $\{V = 8\}$.

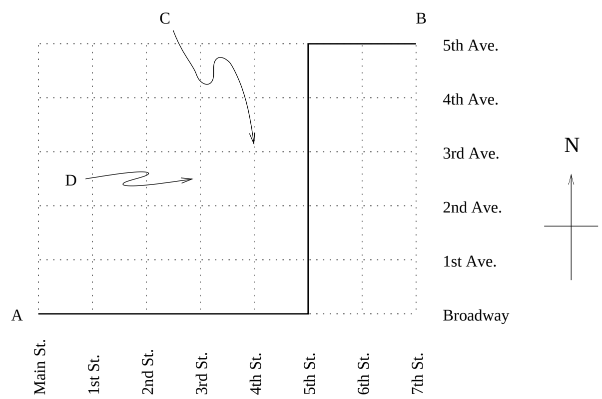
(f) Find and plot the conditional PMF $p_{X|V}(x \mid v)$, for all values.

5. Suppose the waiting time until the next bus at a particular bus stop is exponentially distributed, with parameter $\lambda = \frac{1}{15}$. Suppose that a bus pulls out just as you arrive at the stop. Find the probability that:

(a) You wait more than 15 minutes for a bus.

(b) You wait between 15 and 30 minutes for a bus.

G1†. Clark is a news reporter who lives in Metropolis. Every morning he drives from his apartment at point A to The Daily Planet office at point B for work. He travels only in the north and east directions, and he chooses each path with equal probability. The heavy line in the diagram indicates one such valid path.



(a) Determine the probability that the path in the diagram is the one he actually picked this morning.

(b) There is a phone booth at point C, the intersection of 4th St. and 3rd Ave. Find the probability that he drove past this phone booth on his way to work this morning.

(c) Find the probability that he drove past the phone booth, given that he was seen on the portion of 3rd St. between 2nd and 3rd Ave. (point D).

(d) Repeat part (b) for the general case where his apartment is located at the intersection of Main and Broadway, The Daily Planet is located at the intersection of wth St. and hth Ave., and the phone booth is located at the intersection of xth St. and yth Ave. Assume the same numbering scheme for the streets. (Note that this problem is of interest only if $x \leq w$ and $y \leq h$.) Does your answer make sense for the special cases

i. $w = 0$, *i.e.*, the office is on Main St.; and

ii. $h = 0$, *i.e.*, the office is on Broadway?

†Required for 6.431; optional for 6.041