MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Electrical Engineering & Computer Science
6.041/6.431: Probabilistic Systems Analysis
(Fall 2010)

Recitation 13 Solutions
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1. (a) We begin by writing the definition for $\mathbf{E}[Z \mid X, Y]$

$$\mathbf{E}[Z \mid X = x, Y = y] = \sum_z z p_{Z|X,Y}(z \mid x, y)$$

Since $\mathbf{E}[Z \mid X, Y]$ is a function of the random variables X and Y, and is equal to $\mathbf{E}[Z \mid X = x, Y = y]$ whenever $X = x$ and $Y = y$, which happens with probability $p_{X,Y}(x, y)$, using the expected value rule, we have

$$\begin{aligned} \mathbf{E}\big[\mathbf{E}[Z \mid X, Y]\big] &= \sum_x \sum_y \mathbf{E}[Z \mid X = x, Y = y] p_{X,Y}(x, y) \\ &= \sum_x \sum_y \sum_z z p_{Z|X,Y}(z \mid x, y) p_{X,Y}(x, y) \\ &= \sum_x \sum_y \sum_z z p_{X,Y,Z}(x, y, z) \\ &= \mathbf{E}[Z] \end{aligned}$$

(b) We start with the definition for $\mathbf{E}[Z \mid X, Y]$ which is a function of the random variables X and Y, and is equal to $\mathbf{E}[Z \mid X = x, Y = y]$ whenever $X = x$ and $Y = y$, so

$$\mathbf{E}[Z \mid X = x, Y = y] = \sum_z z p_{Z|X,Y}(z \mid x, y)$$

Proceeding as above, but conditioning on the event $X = x$, we have

$$\begin{aligned} \mathbf{E}\big[\mathbf{E}[Z \mid X, Y = y] \mid X = x\big] &= \sum_y \mathbf{E}[Z \mid X = x, Y = y] p_{Y|X}(y \mid x) \\ &= \sum_y \sum_z z p_{Z|X,Y}(z \mid x, y) p_{Y|X}(y \mid x) \\ &= \sum_y \sum_z z p_{Y,Z|X}(y, z \mid x) \\ &= \mathbf{E}[Z \mid X = x] \end{aligned}$$

Since this is true for all possible values of x, we have $\mathbf{E}\big[\mathbf{E}[Z \mid Y, X] \mid X\big] = \mathbf{E}[Z \mid X]$.

(c) We take expectations of both sides of the formula in part (b) to obtain

$$\mathbf{E}\big[\mathbf{E}[Z \mid X]\big] = \mathbf{E}\big[\mathbf{E}\big[\mathbf{E}[Z \mid X, Y] \mid X\big]\big].$$

By the law of iterated expectations, the left-hand side above is $\mathbf{E}[Z]$, which establishes the desired result.

2. Let Y be the length of the piece after we break for the first time. Let X be the length after we break for the second time.

(a) The law of iterated expectations states:

$$\mathbf{E}[X] = \mathbf{E}[\mathbf{E}[X|Y]]$$

We have $\mathbf{E}[X|Y] = \frac{Y}{2}$ and $E[Y] = \frac{l}{2}$. So then:

$$\mathbf{E}[X] = \mathbf{E}[\mathbf{E}[X|Y]] = \mathbf{E}[Y/2] = \frac{1}{2}\mathbf{E}[Y] = \frac{1}{2}\frac{l}{2} = \frac{l}{4}$$

(b) We use the Law of Total Variance to find $\text{var}(X)$:

$$\text{var}(X) = \mathbf{E}[\text{var}(X \mid Y)] + \text{var}(\mathbf{E}[X \mid Y]).$$

Recall that the variance of a uniform random variable distributed over $[a, b]$ is $(b-a)^2/12$. Since Y is uniformly distributed over $[0, \ell]$, we have

$$\begin{aligned} \text{var}(Y) &= \frac{\ell^2}{12}, \\ \text{var}(X \mid Y) &= \frac{Y^2}{12}. \end{aligned}$$

We know that $\mathbf{E}[X \mid Y] = Y/2$, and so

$$\text{var}(\mathbf{E}[X \mid Y]) = \text{var}(Y/2) = \frac{1}{4}\text{var}(Y) = \frac{\ell^2}{48}.$$

Also,

$$\begin{aligned} \mathbf{E}[\text{var}(X \mid Y)] &= \mathbf{E}\left[\frac{Y^2}{12}\right] \\ &= \int_0^\ell \frac{y^2}{12} f_Y(y)dy \\ &= \frac{1}{12} \cdot \frac{1}{\ell}\int_0^\ell y^2 dy \\ &= \frac{\ell^2}{36}. \end{aligned}$$

Combining these results, we obtain

$$\text{var}(X) = \mathbf{E}[\text{var}(X \mid Y)] + \text{var}(\mathbf{E}[X \mid Y]) = \frac{\ell^2}{36} + \frac{\ell^2}{48} = \frac{7\ell^2}{144}.$$

3. Let X_i denote the number of widgets in the i^{th} box. Then $T = \sum_{i=1}^N X_i$.

$$\begin{aligned} \mathbf{E}[T] &= \mathbf{E}[\mathbf{E}[\sum_{i=1}^N X_i|N]] \\ &= \mathbf{E}[\sum_{i=1}^N \mathbf{E}[X_i|N]] \\ &= \mathbf{E}[\sum_{i=1}^N \mathbf{E}[X]] \\ &= \mathbf{E}[X] \cdot \mathbf{E}[N] = 100. \end{aligned}$$

and,

$$\begin{aligned}
\text{var}(T) &= \mathbf{E}\left[\text{var}(T|N)\right] + \text{var}\left(\mathbf{E}[T|N]\right) \\
&= \mathbf{E}\left[\text{var}\left(\sum_{i=1}^{N} X_i|N\right)\right] + \text{var}\left(\mathbf{E}\left[\sum_{i=1}^{N} X_i|N\right]\right) \\
&= \mathbf{E}[N\text{var}(X)] + \text{var}(N\mathbf{E}[X]) \\
&= (\text{var}(X))\mathbf{E}[N] + (\mathbf{E}[X])^2\,\text{var}(N) \\
&= 16 \cdot 10 + 100 \cdot 16 = 1760.
\end{aligned}$$