Show All Answers Hide All Answers

6.02 Practice Problems: Frequency Response of LTI Systems & Filters

Note: In these problems, we sometimes refer to H(Ω) as H($e^{j\Omega}$). The reason is that in some previous terms we used the latter convention, and it's too painful to redraw all the figures, which embed the latter notation. Anyway, there should be no confusion because the convention should be apparent from the context.

Problem 1.

Give an expression for the magnitude of a complex exponential with frequency φ, i.e., $|e^{j\varphi}|$.

Hide Answer

$|e^{j\varphi}|$ = |cos(φ) + j*sin(φ)| = sqrt($\cos^2(\varphi)$ + $\sin^2(\varphi)$) = 1

Problem 2.

A. Prove the validity of the following formula, often referred to as the *finite sum formala*:

$$\sum_{n=0}^{N-1} \alpha^n = \begin{cases} N & \alpha = 1 \\ \dfrac{1-\alpha^N}{1-\alpha} & \text{for any complex number } \alpha \neq 1 \end{cases}$$

Hide Answer

For α ≠ 1, do the following:
1. expand the sum
2. multiply expansion by (1–α)/(1–α)
3. simplify the numerator

You can also get the same answer by observing that the summation is a geometric series.

B. Use the finite sum formula to show:

$$\sum_{n=\langle N \rangle} e^{jk\frac{2\pi}{N}n} = \begin{cases} N & k = 0, \pm N, \pm 2N, \ldots \\ 0 & \text{otherwise} \end{cases}$$

Hide Answer

Simply apply the finite sum formula and use the fact that $e^{j\pi} = -1$.

Problem 3.

Consider an LTI system characterized by the unit-sample response h[n].

A. Give an expression for the frequency response of the system $H(e^{j\Omega})$ in terms of $h[n]$.

Hide Answer

$H(e^{j\Omega}) = \Sigma_m h[m] e^{-j\Omega m}$.

B. If $h[0]=1$, $h[1]=0$, $h[2]=1$, and $h[n]=0$ for all other n, what is $H(e^{j\Omega})$?

Hide Answer

Simply substitute in the formula above.
$H(e^{j\Omega}) = 1 + e^{-2j\Omega}$.

C. Let $h[n]$ be defined as in part B and $x[n] = \cos(\varphi n)$. Is there a value of φ such that $y[n]=0$ for all n and $0 \le \varphi \le \pi$?

Hide Answer

We need to find the Omega at which H goes to zero.
$H(e^{j\Omega})=0$ when $\varphi = \pi/2$.

D. Let $h[n]$ be defined as in part B. Find the maximum magnitude of $y[n]$ if $x[n] = \cos(\pi n/4)$.

Hide Answer

The max. magnitude of y is the max. magnitude of x (1 since x is a cosine) multiplied by the maginitude of the freq. response.
$|H(e^{j\pi/4})| = |1 + e^{-j\pi/2}| = |1 - j| = \text{sqrt}(2)$.

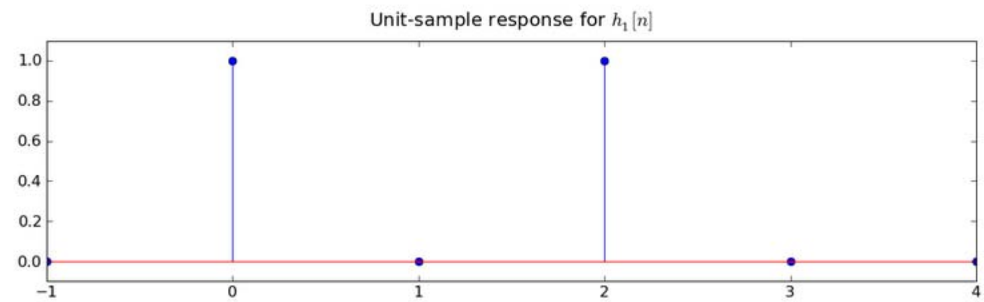
E. Let $h[n]$ be defined as in part B. Find the maximum magnitude of $y[n]$ if $x[n] = \cos(-(\pi/2)n)$.

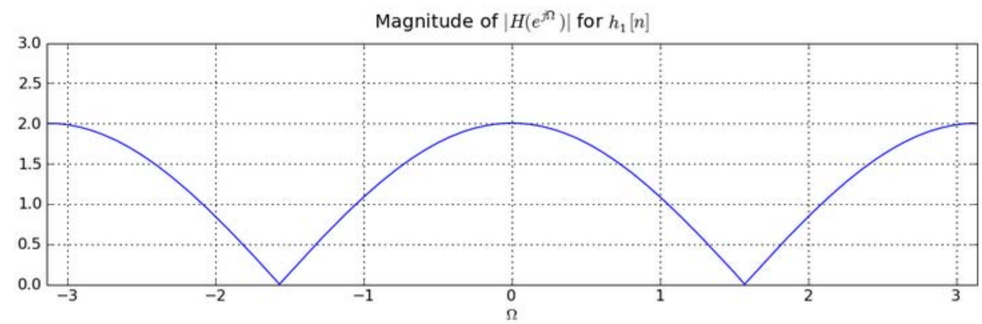
Hide Answer

$|H(e^{j\pi/2})| = |1 + e^{-j\pi}| = |1 - 1| = 0$.

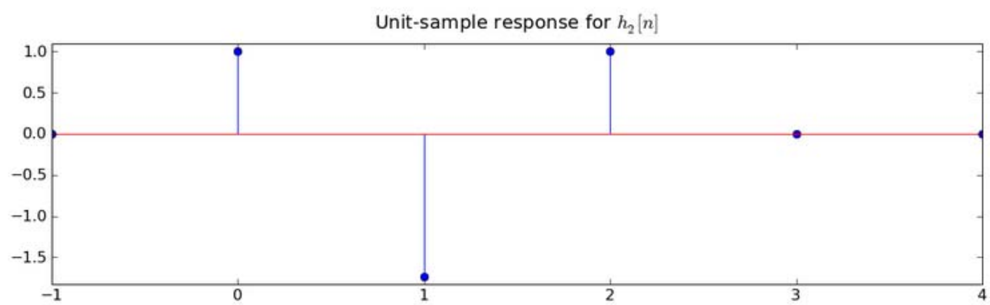
Problem 4.

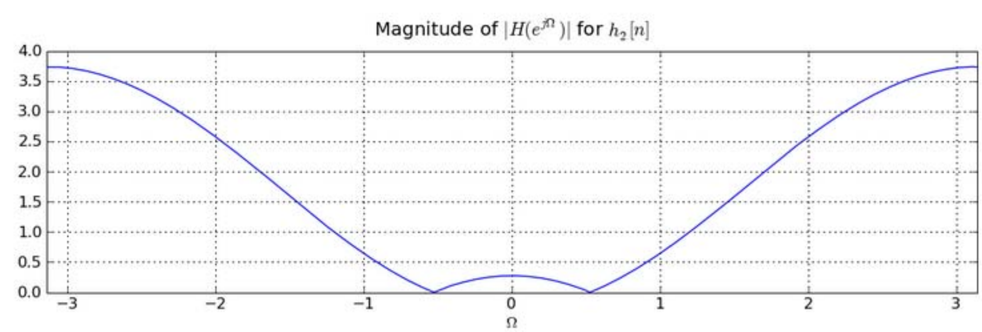
In answering the questions below, please consider the unit sample response and frequency response of two filters, H_1 and H_2, plotted below.





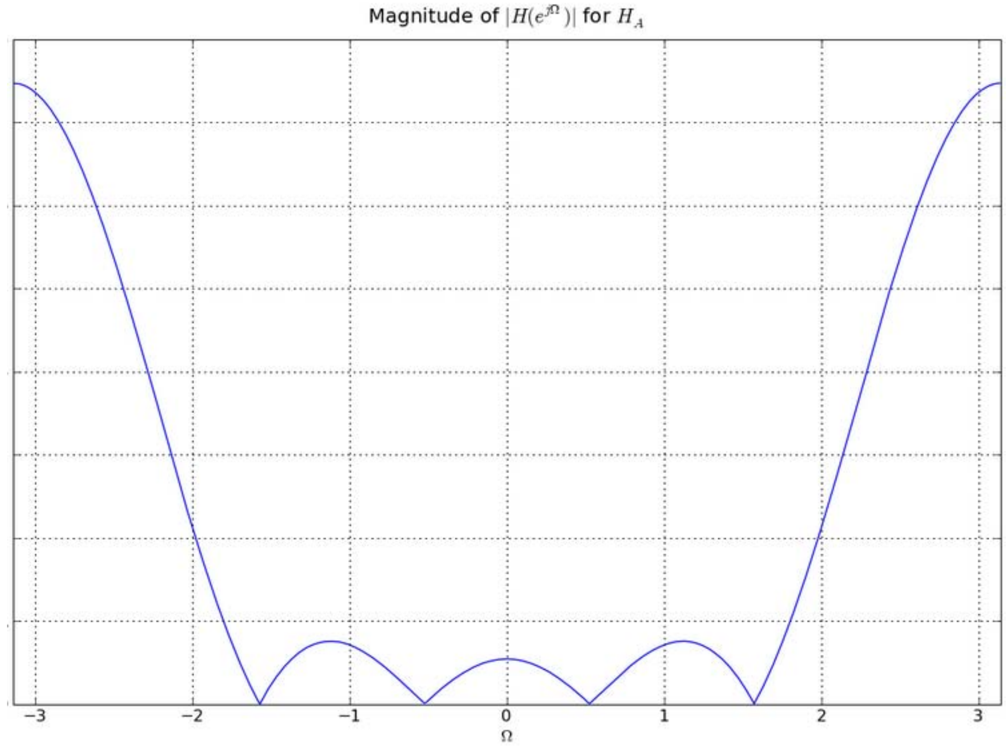
Note: the only nonzero values of unit sample response for H_1 are : $h_1[0] = 1$, $h_1[1]=0$, $h_1[2]=1$.

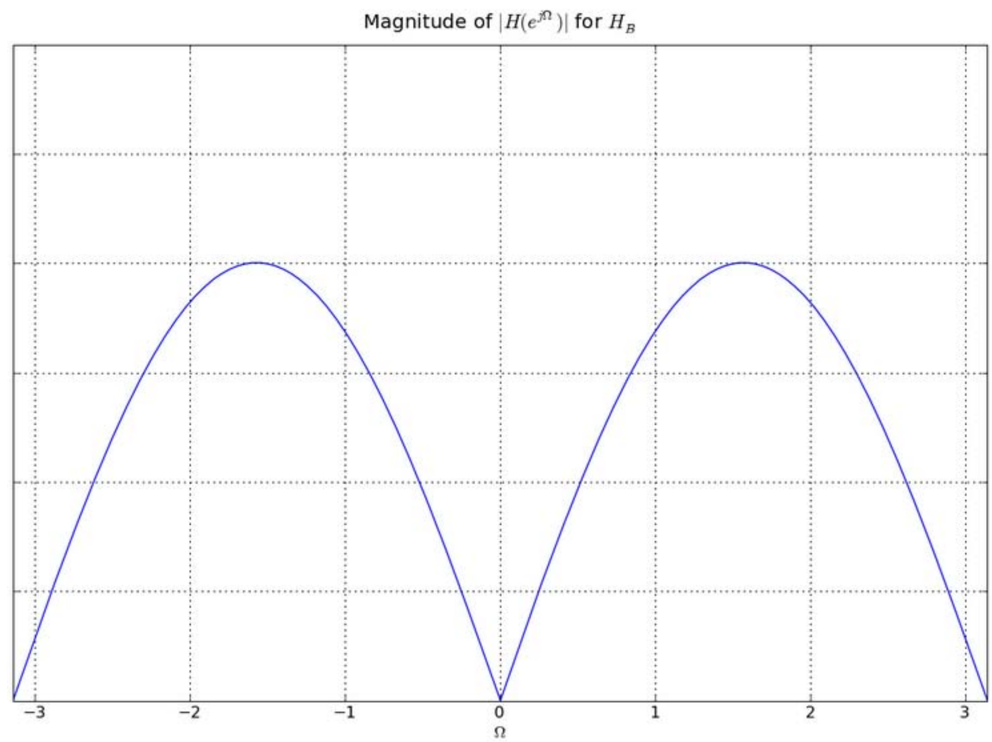


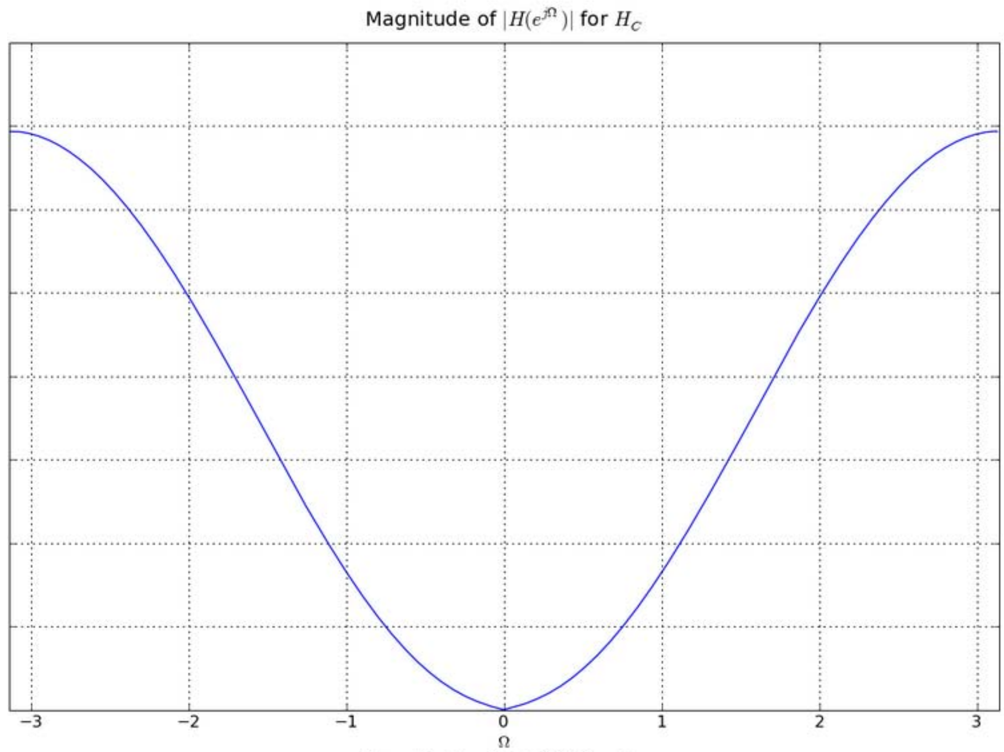


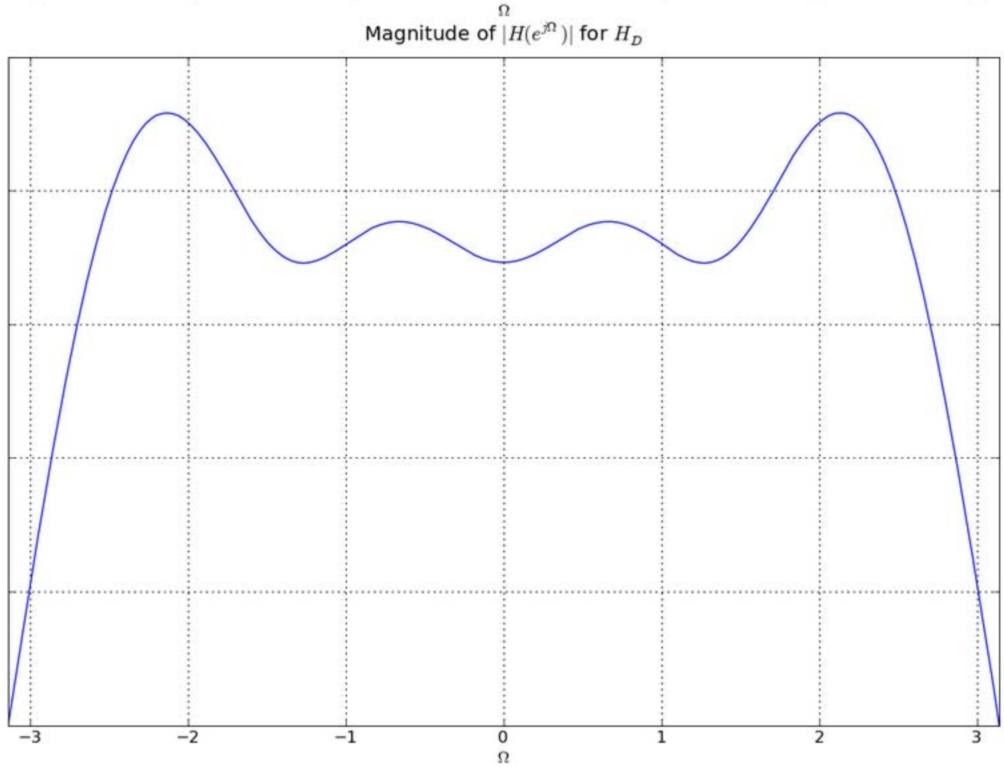
Note, the only nonzero values of unit sample response for H_2 are : $h_2[0] = 1$, $h_2[1]=-\text{sqrt}(3)$, $h_2[2]=1$.

In answering the several parts of this review question consider four linear time-invariant systems, denoted A, B, C, and D, each characterized by the magnitude of its frequency response, $|H_A(e^{j\Omega})|$, $|H_B(e^{j\Omega})|$, $|H_C(e^{j\Omega}\})|$, and $|H_D(e^{j\Omega})|$ respectively, as given in the plots below. This is a review problem, not an actual exam question, so similar concepts are tested multiple times to give you practice









A. Which frequency response (A, B, C or D) corresponds to a unit sample response given by

$h[n] = \alpha\ \delta[n] - h_1[n]$

and what is the numerical value of $|\alpha|$?

Hide Answer

The frequency response is a linear operation. So to get the frequency response of the sum of two unit sample responses, we can simply add the frequency responses for the original ones. The frequency response for h1 is already given in the figure above and can be computed to give $H(e^{j\Omega}) = 1 + e^{-2j\Omega}$.

(Problem 3 part b). The frequency response of

$\alpha\ \delta[n]$ is simply the constant α . If we add the two responses and compute the magnitude of the result, we see that it must have exactly 2 extrema, one each at $\pm\pi/2$. So it must be H_B as that is the only frequency response that has extrema exactly at $\pm\pi/2$

To determine α we look at the resulting response at the point where Omega is zero.

$|\alpha|=2$ as $H_1(e^{j0})=2$ but $H_B(e^{j0})=0$.

B. Which frequency response (A, B, C or D) corresponds to a unit sample response given by

$h[n] = \Sigma_m h_1[m]h_2[n-m]$ for $m = 0$ to n

and what are the numerical values of $h[2]$, $h[3]$ and $H(e^{j0})$?

Hide Answer

Apply the result that convolution in time domain implies multiplication in the frequency domain. Hence, it must be H_A since $H_1(e^{j\Omega})H_2(e^{j\Omega}) = H(e^{j\Omega})$, and therefore $|H(e^{j\Omega})|=0$ whenever $H_1(e^{j\Omega})=0$ or $H_2(e^{j\Omega})=0$.

$H(e^{j0}) = H_1(e^{j0})H_2(e^{j0}) = 2(2 - \text{sqrt}(3)) = 4 - 2\text{sqrt}(3)$

$h[n] = [1,0,1]*[1,-\text{sqrt}(3),1]$ so $h[2] = 2$ and $h[3] = -\text{sqrt}(3)$.

C. Which frequency response (A, B, C or D) corresponds to a unit sample response given by

$h[n] = \alpha\ \delta[n] - \Sigma_m h_1[m]h_2[n-m]$ for $m = 0$ to n

and what is the numerical value of $|\alpha|$?

Hide Answer

H_A is the frequency response for H_1*H_2 as seen in the previous question. The frequency response H here is $H=\alpha - H_A$ using the same linearity argument as earlier. Hence H must have the same values at all those points where H_A goes to zero. Since H_A has 4 zeros, there are 4 such points where H must have the same values . In both H_B and H_D we can find 4 points that have the same value on the y-axis. But only in H_D the x-coordinates of these 4 points match up to the x-coordinates of the zeroes of H_A. So, H_D must be the solution.

The resulting spectra in H_D goes to zero at +/- PI. Since $|H_A(e^{j\pi})| = |H_1(e^{j\pi})||H_2(e^{j\pi})|$, $|\alpha| = 2(2 + \text{sqrt}(3)) = 4 + 2\text{sqrt}(3)$. This is similar to the way alpha was computed in part A.

D. Which frequency response (A, B, C or D) corresponds to a unit sample response given by

$h[n] = \alpha\ \delta[n] - h_2[n]$

and what is the numerical value of $|\alpha|$?

Hide Answer

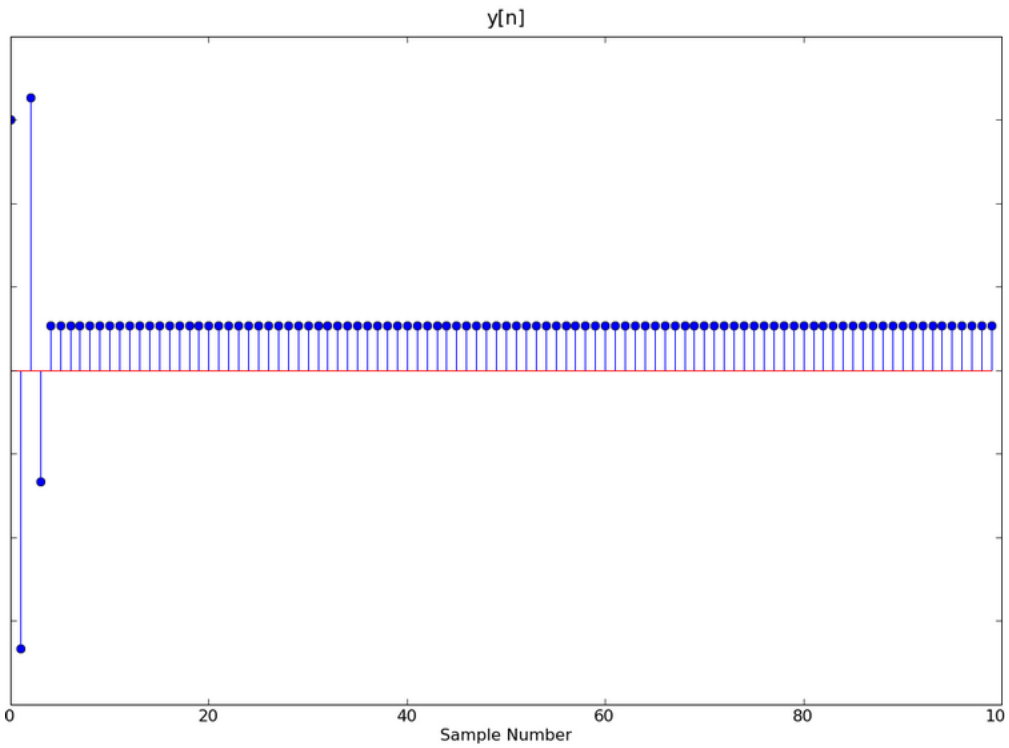
Must be H_C by elimination but also because none of the other frequency repsonse could be generated by simply adding a real constant alpha to the spectrum of $H_2(e^{j\Omega})$.

$H_C(e^{j0}) = 0$ so $|\alpha| = |H_2(e^{j0})| = 2 - \text{sqrt}(3)$.

E. Suppose the input to each of the above four systems is

$x[n]=0$ for $n < 0$ and
$x[n] = \cos(n\pi/6) + \cos(n\pi/2) + 1.0$ for $n \geq 0$

Which system (A, B, C or D) produced an output, $y[n]$ below, and what is the value of $y[n]$ for $n > 10$?



Hide Answer

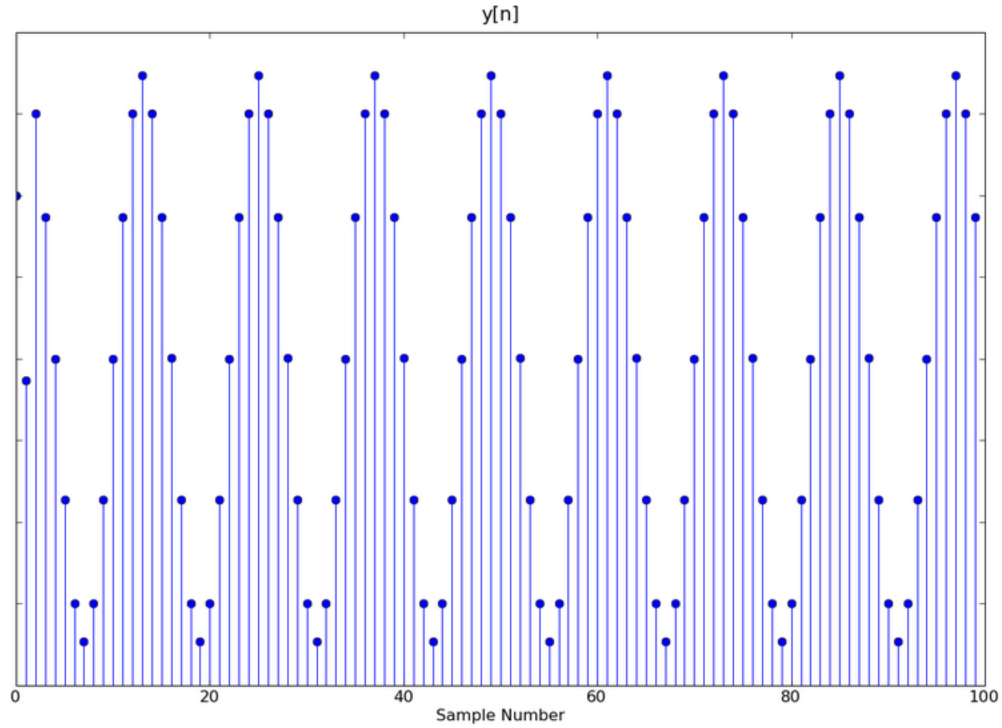
The steady state response to this system is a constant. Hence the frequency response must be zero at all other frequencies. Must be H_A, as $|H_A(e^{j\Omega})| = 0$ for $\Omega=\pm\pi/6$ and $\Omega=\pm\pi/2$.

$y[n]$ for $n > 10 = |H_A(e^{j0})| * 1 = 4 - 2\text{sqrt}(3)$

F. Suppose the input to each of the above four systems is

$x[n]=0$ for $n < 0$ and
$x[n] = \cos(n\pi/6) + \cos(n\pi/2) + 1.0$ for $n \geq 0$

Which system (H_1 or H_2) produced an output, $y[n]$ below, and what is the value of $y[22]$?



Hide Answer

The resulting output signal has a period of 12 samples implying it's discrete frequency is PI/6. Hence it must be H_1 since the H_2 system would eliminate $\cos(\pi/6)$. This is because h2 is a 3-tap filter with nulling frequency at PI/6 (since h[2]=-2cos(Phi) and we know that h[2] is sqrt(3), Phi must be PI/6). Further more the resulting output signal doesn't have the PI/2 frequency component anymore. This is exactly what h1 eliminates if you look at it's frequency response. So, the system must be H1.

The output will eventually be a cosine added to $H_1(e^{j0})*1$.

First compute some useful H's from the given h[n]:

$$H(e^{j\Omega}) = 1 + e^{-j2\Omega}$$
$$H(e^{j0}) = 1 + 1 = 2$$
$$H(e^{j(-\pi/6)}) = 1 + e^{j\pi/3}$$
$$H(e^{j(\pi/6)}) = 1 + e^{-j\pi/3}$$
$$H(e^{j(-\pi/2)}) = 1 + e^{j\pi} = 1 - 1 = 0$$
$$H(e^{j(\pi/2)}) = 1 + e^{-j\pi} = 1 - 1 = 0$$

Now we can plug those into our equation for y[n] that uses the spectral coefficients for x[n] and the frequency response H:

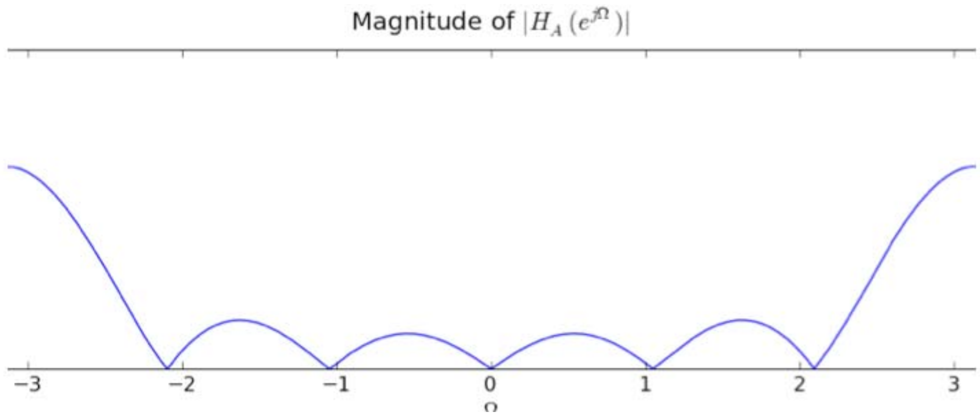
$$\begin{aligned} y[n] &= \Sigma_k a_k H(e^{jk(2\pi/N)})e^{jk(2\pi/N)n} \\ &= 2 + (1/2)H(e^{j(-\pi/6)})e^{j(-\pi/6)n} + (1/2)H(e^{j(\pi/6)})e^{j(\pi/6)n} \\ &= 2 + (1/2)(1 + e^{j\pi/3})e^{j(-\pi/6)n} + (1/2)(1 + e^{-j\pi/3})e^{j(\pi/6)n} \\ &= 2 + (1/2)e^{j(-\pi/6)n} + (1/2)e^{-j((\pi/6)n - \pi/3)} + (1/2)e^{j(\pi/6)n} + (1/2)e^{j((\pi/6)n - \pi/3)} \\ &= 2 + \cos((\pi/6)n) + \cos((\pi/6)n - \pi/3) \end{aligned}$$

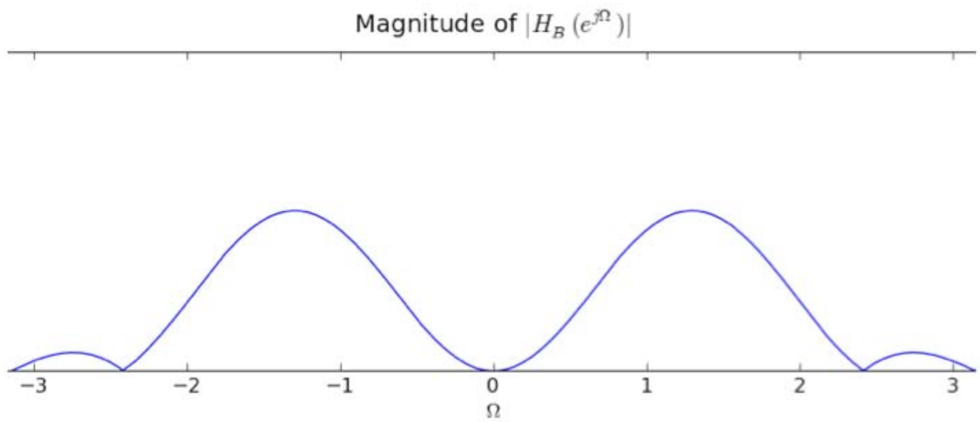
So

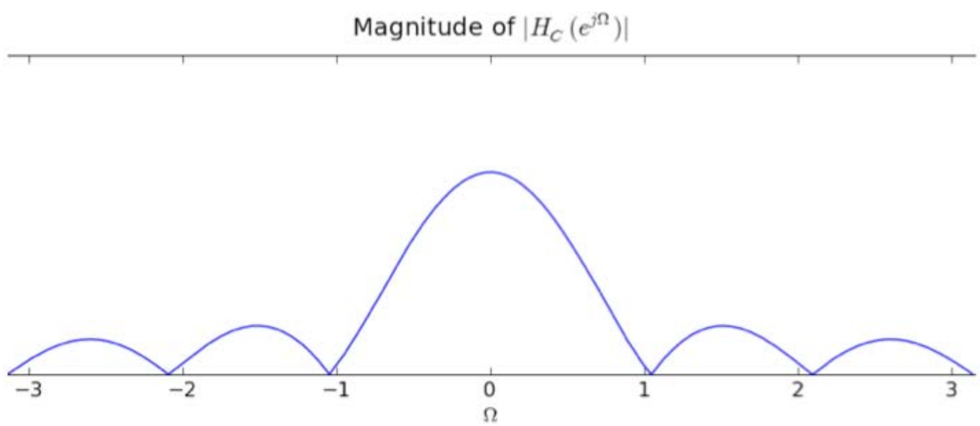
$$\begin{aligned} y[22] &= 2 + \cos(22\pi/6) + \cos(22\pi/6 - \pi/3) \\ &= 2 + 0.5 - 0.5 \\ &= 2 \end{aligned}$$

Problem 5.

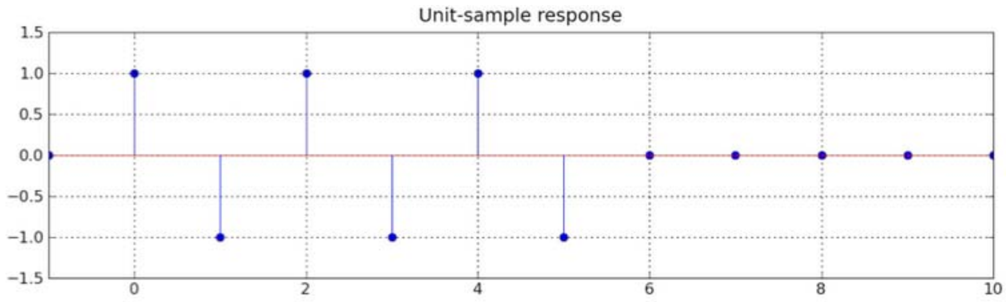
In answering the several parts of this question, consider three linear time-invariant filters, denoted A, B, and C, each characterized by the magnitude of their frequency responses, $|HA(e^{j\Omega})|$, $|HB(e^{j\Omega})|$, $|HC(e^{j\Omega})|$, respectively, as given in the plots below.







A. Which frequency response (A, B, or C) corresponds to the following unit sample response, and what is $\max_\Omega |H(e^{j\Omega})|$ for your selected filter? Please justify your selection.



Hide Answer

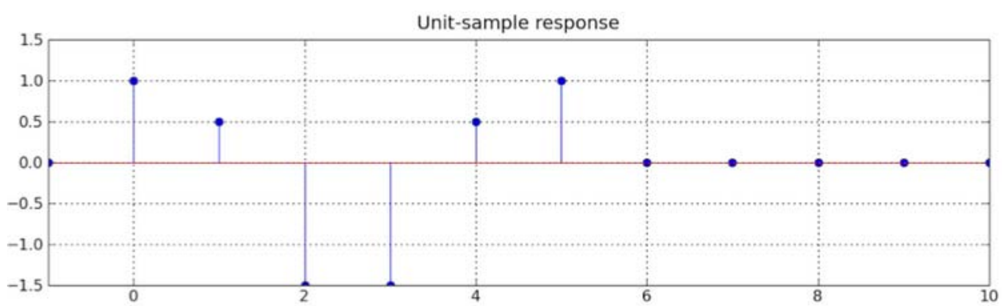
Plan of attack: evaluate $|H(e^{j\Omega})|$ at several frequencies and see what we find.

$\Omega=0$: $|H(e^{j0})| = |\Sigma h[n](1)| = 0$
$\Omega=\pi$: $|H(e^{j\pi})| = |\Sigma h[n](-1)^n| = 6$

The only plot with a frequency response of 0 at frequency 0 and something non-zero at frequency π is the one for filter A.

Looking at the frequency response for A, it has its largest magnitude at ±π, which we calculated above to be 6.

B. Which frequency response (A, B, or C) corresponds to the following unit sample response, and is $\max_\Omega |H(e^{j\Omega})| > 6$ for your selected filter? Please justify your answers.



Hide Answer

Repeating the strategy of part A:

$\Omega=0$: $|H(e^{j0})| = |\Sigma h[n]| = 1 + .5 - 1.5 - 1.5 + .5 + 1 = 0$
$\Omega=\pi$: $|H(e^{j\pi})| = |\Sigma h[n](-1)^n| = 1 -.5 - 1.5 + 1.5 +.5 - 1 = 0$

The only plot with a frequency response of 0 at frequencies 0 and π is the one for filter B.

The maximum magnitude of the frequency response is given by:

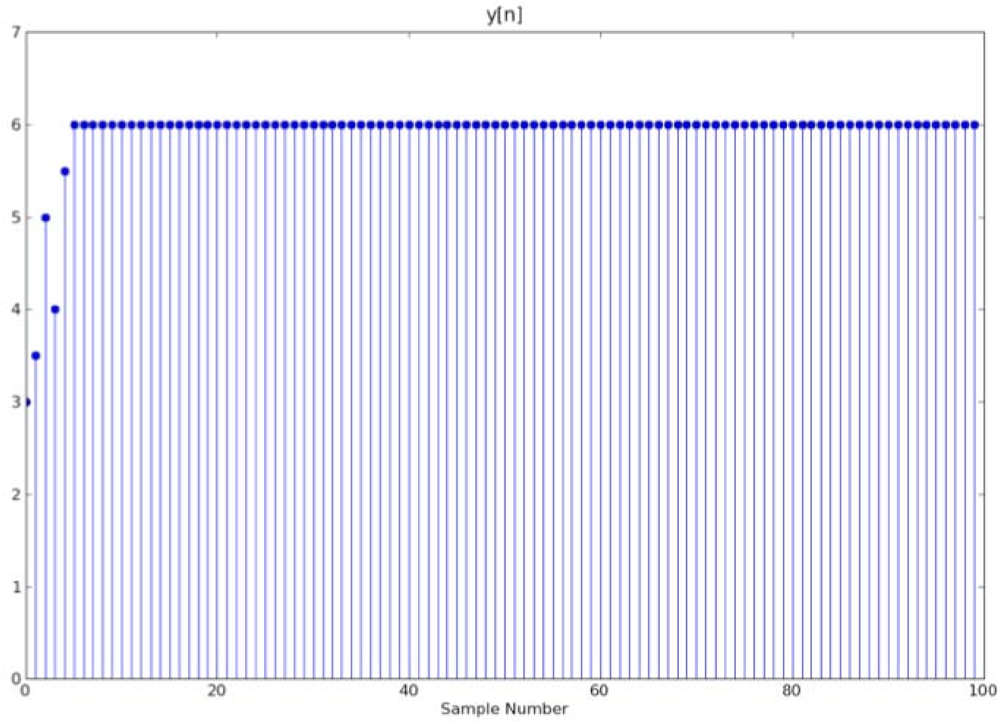
$$|H(e^{j\Omega})| = |\Sigma_n h[n]e^{-j\Omega n}| \leq \Sigma_n |h[n]|(1) \leq 6.$$

So, no, it's not bigger than 6.

C. Suppose the input to each of the above three filters is $x[n] = 0$ for $n < 0$ and for $n \geq 0$ is

$$x[n] = \cos((\pi/3)n) + \cos(\pi n) + 1.0$$

Which filter (A, B,or C) produced the output, $y[n]$ below, and what is $\max_\Omega |H(e^{j\Omega})|$ for your selected system?



Hide Answer

The output reaches a steady state at `y[n] = 6`, with no changes in value after that point. So sinusoidal components of `x[n]` must have been filtered out, i.e., $H(e^{j\pi/3}) = 0$ and $H(e^{j\pi}) = 0$.

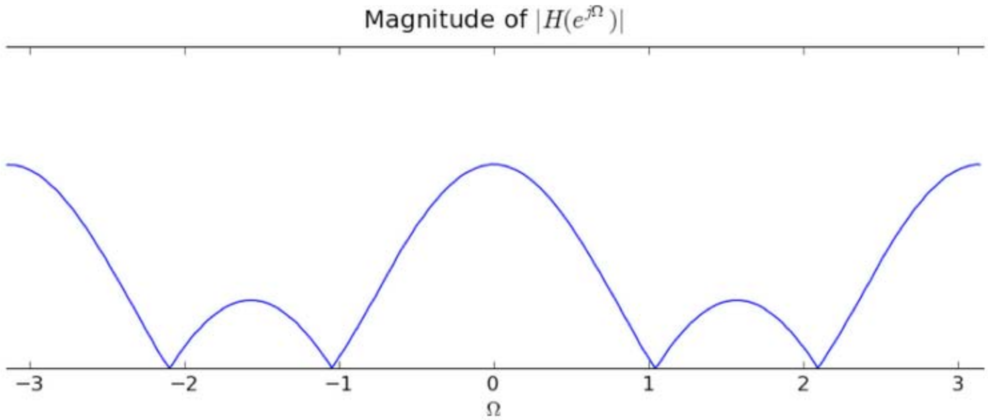
The only other component of `x[n]` is the constant 1, so that means that $H(e^{j0}) = 6$ since constant inputs have frequency 0 and the constant output from the figure stays at 6.

The only plot that satisfies these conditions is C. From the plot of C's frequency response, the maximum magnitude occurs at frequency 0 and has the value 6 as we calculated above.

D. Six new filters were generated using the unit sample responses of filters A, B and C, denoted `hA[n]`, `hB[n]`, and `hC[n]` respectively. The unit sample responses of the new filters were generated in the following way:

- `h1[n] = hA[n] + hB[n]`. Filters A and B in parallel.
- `h2[n] = hA[n] + hC[n]`. Filters A and C in parallel.
- `h3[n] = hB[n] + hD[n]`. Filters B and C in parallel.
- `h4[n] = convolve(hA,hB)`. Filters A and B in series.
- `h5[n] = convolve(hA,hC)`. Filters A and C in series.
- `h6[n] = convolve(hB,hC)`. Filters B and C in series.

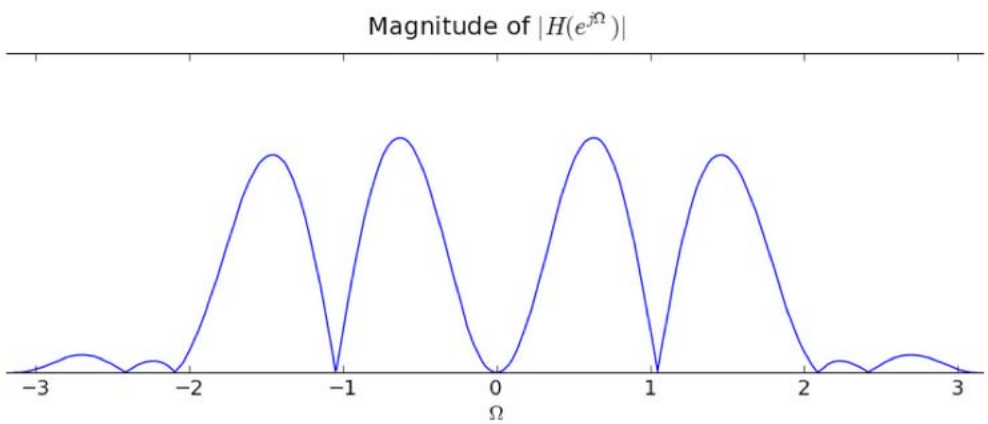
Which of the six new filters has the frequency response plotted below?



Hide Answer

Recall, that in parallel, frequency responses are added together. Only H_C in non-zero at frequency 0 and only H_A is non-zero at frequency π. So plot must correspond to new filter 2, which is filters A and C in parallel.

Which of the six new filters from the previous question has the frequency response plotted below?



Hide Answer

Recall, that in series, frequency responses are multiplied together. Looking at the zeros in the plot above, the combination could have only come from two filters in series. Comparing the size of the response near frequency 0 with the response near frequency π, it must be filter B in series with filter C, i.e., new filter 6.