Problem #1

GIVEN: "Whatever" vector, **W**(x,y,z) = $\mathbf{i}_x \sin(y) + \mathbf{i}_y\, y$ (the **i**'s are unit vectors).
FIND: (a) If **W** were an electric displacement vector, **D**, what would be the corresponding charge density, ρ?
WORK:
Gauss' law for electric fields:

$$\left(\partial_x \rightarrow \frac{\partial}{\partial x},\ \partial_y \rightarrow \frac{\partial}{\partial y},\ \partial_z \rightarrow \frac{\partial}{\partial z} \right)$$

"Divergence"

$$\boxed{\nabla \cdot \mathbf{D} = \rho = \partial_x D_x + \partial_y D_y + \partial_z D_z = \partial_y D_y = 1 \ [\mathrm{C/m^3}]}$$

FIND: (b) If **W** were a magnetic field vector, **H**, what would be the corresponding current density, **J**?
WORK:
Ampère's law in the *steady state*:

"curl"

$$\nabla \times \vec{H} = \vec{J} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial_x & \partial_y & \partial_z \\ H_x & H_y & H_z \end{vmatrix} = \begin{array}{l} \hat{x}(\partial_y H_z - \partial_z H_y) \\ -\hat{y}(\partial_x H_z - \partial_z H_x) \\ +\hat{z}(\partial_x H_y - \partial_y H_x) \end{array}$$

$$= \hat{z}(\partial_x H_y - \partial_y H_x) = \hat{z}\left[\partial_x y - \partial_y \sin(y)\right] = \boxed{-\hat{z}\cos(y) = \vec{J}}$$

FIND: (c) Does the magnetic field determined in (b) satisfy all of Maxwell's equations? If not, which one is violated?
WORK: The constitutive relation for the magnetic field quantities is assumed linear, **B**=μ**H**, where μ is also assumed to be a constant. In this case, we immediately know that $\nabla \cdot \mathbf{B} = \nabla \cdot (\mu \mathbf{H}) = \mu \nabla \cdot \mathbf{W} \neq 0$ from part (a), in which case Gauss' law is violated.

In an Ohmic medium with the constitutive relation, **J** = σ**E**, with the conductivity, σ, a constant, then **E** = **J**/σ, and $\nabla \times \mathbf{J} = -\mathbf{i}_x \sin(y) \neq 0$, in which case Faraday's law of induction ($\nabla \times \mathbf{E} = -\partial \mathbf{B}/\partial t$) would be violated for the steady state.

2.2 Problem 2

GIVEN

If the electric field is $E(t) = \Re\left\{\tilde{E}e^{j\omega t}\right\}$, where $\tilde{E}$ is a phasor, then what is $E(t)$ if

FIND

(a) $\tilde{E} = 1 - j$

WORK

$e^{j\omega t} = \cos(\omega t) + j\sin(\omega t)$ by Euler's theory. If $\tilde{E} = \tilde{E}_r + j\tilde{E}_i$, where $\tilde{E}_r$ and $\tilde{E}_i$ are both real quantities, then

$$\tilde{E}e^{j\omega t} = \tilde{E}_r\cos(\omega t) - \tilde{E}_i\sin(\omega t) + j\left(\tilde{E}_i\cos(\omega t) + \tilde{E}_r\sin(\omega t)\right), \qquad (2.1)$$

the real part of which is

$$E(t) = \Re(\tilde{E}e^{j\omega t}) = \tilde{E}_r\cos(\omega t) - \tilde{E}_i\sin(\omega t). \qquad (2.2)$$

Alternatively, $\tilde{E}$ may be expressed as the product of a magnitude and an exponential,
$\tilde{E} = \left|\tilde{E}\right| e^{j\phi}$ where $\phi = \arctan\left(\frac{\tilde{E}_i}{\tilde{E}_r}\right)$. Then

$$\tilde{E}e^{j\omega t} = \left|\tilde{E}\right| e^{j(\omega t+\phi)} = \left|\tilde{E}\right| \left(cos(\omega t+\phi) + j\sin(\omega t+\phi)\right), \qquad (2.3)$$

the real part of which is

$$E(t) = \Re(\tilde{E}e^{j\omega t}) = \left|\tilde{E}_r\right| \cos(\omega t+\phi). \qquad (2.4)$$

Now, turn the crank and apply trigonometric identities to get more meaningful answers. In (a), $\tilde{E}_r = 1$ and $\tilde{E}_i = -1$, so

$\boxed{E(t) = \cos(\omega t) + \sin(\omega t) = \sqrt{2}\sin(\omega t + 45^\circ)}$

Also, the *phase angle*, ϕ, is $\arctan(-1) = -45^\circ$, while the magnitude is $\sqrt{2}$, so

$\boxed{E(t) = \sqrt{2}\cos(\omega t - 45^\circ) = \sqrt{2}\sin\omega t + 45^\circ}$

Either description gives the same result. In this case, the second is a little more straightforward because the magnitude and phase angle of the phasor may be obtained almost by inspection.

FIND

(b) $\tilde{E} = e^{j\pi/4} - 1$

WORK

$\tilde{E} = e^{j\pi/4} - 1 = \sqrt{2}/2\left(1 - \sqrt{2} + j\right)$
$\phi = \arctan\left(\frac{1-\sqrt{2}/2}{\sqrt{2}/2}\right) = \arctan\left(\frac{-1+\sqrt{2}/2}{\sqrt{2}/2}\right) = -67.5^\circ, 112.5^\circ, ..., n(180^\circ) + 112.5^\circ$
since arctan() is periodic through 180 degrees. By inspection, we know that $e^{j\pi/4} - 1$ lies in the second quadrant of the complex plane, and so we reason that the phase angle is uniquely 112.5 degrees. $\left|\tilde{E}\right| = \sqrt{\tilde{E}_r^2 + \tilde{E}_i^2} = \sqrt{\frac{3}{2} - \sqrt{2} + \frac{1}{2}} = \sqrt{2 - \sqrt{2}} \approx 0.7654$

So

$$\boxed{E(t) = \sqrt{2 - \sqrt{2}}\cos(\omega t + 112.5^\circ) \approx 0.7654\cos(\omega t + 112.5^\circ)}$$

FIND

(c) $\tilde{E} = j\hat{x} + \hat{y}(1 - j)$

WORK

Now, the phasor is a vector quantity, which means that $E(t) \to \vec{E}(t)$. We have to do everything componentwise! (Or we can change the coordinate system....)

Fortunately, we've already done the work for the $\hat{y}$-component in part (a). By inspection, the phase angle for the $\hat{x}$-component is 90°, and the magnitude is one, and $cos(\theta + \pi/2) = -sin(\theta)$. Therefore,

$$\boxed{\vec{E}(t) = -\hat{x}\sin(\omega t) + \hat{y}\sqrt{2}\sin(\omega t + 45^\circ) = \sin(\omega t)\left(-\hat{x} + \hat{y}\right) + \cos(\omega t)\hat{y}}$$

This is an example of an *elliptically-polarized* wave.

FIND

$\tilde{E}$ if $E(t) = \hat{x}\cos(\omega t) + \hat{y}\sin(\omega t + \pi/4)$.

WORK

The $\hat{y}$-component is identified immediately as the same result of the inverse problem in part(a), so again, $\tilde{E}_y = 1 - j$. By inspection, the $\hat{x}$ component has a phase angle of $\phi = 0$ and unity magnitude, so $\tilde{E}_x = 1$. Then

$$\boxed{\tilde{E} = \hat{x} + \hat{y}\frac{(1-j)}{\sqrt{(2)}}}$$

Useful Matlab Commands

```
Ep = exp(j*pi/4)-1 %Define complex phasor. You can use "i" instead of "j".
angle(Ep) %Get phase angle of phasor.
angle(Ep)*180/pi %Get phase angle of phasor in degrees.
abs(Ep) %Get magnitude of phasor.
%You could also do symbolic manipulation...
syms w t
real( Ep*exp(j*w*t) )
%...but you wouldn't like the result (try it).
%Plot the phasor in the complex plane.
polar([0 angle(Ep)], [0 abs(Ep)])
```

2.3 Problem 3

GIVEN

$\vec{H} = \hat{x}\sin(10^7\pi t - 0.2z) + \hat{y}\cos\left(10^7\pi t - 0.2z - \frac{5}{2}\pi\right)$

2.3.1 Part a

FIND

Frequency.

WORK

First, for clarity, $\cos\left(10^7\pi t - 0.2z - \frac{5}{2}\pi\right) = \cos(10^7\pi t - 0.2z)\cos\left(\frac{5\pi}{2}\right) + \sin(10^7\pi t - 0.2)\sin\left(\frac{5\pi}{2}\right)$, and since $\cos(5\pi/2) = \cos(\pi/2) = 0$, while $\sin(5\pi/2) = \sin(\pi/2) = 1$, $\cos\left(10^7\pi t - 0.2z - \frac{5}{2}\pi\right) = \sin\left(10^7\pi t - 0.2z\right)$.

The phase front of the wave is given by the argument of the sinusoidal functions: $10^7\pi t - 0.02z$. This is a forward-propogating wave (increase the time a little bit, and you have to move forward in the $\hat{z}$-direction to catch up with the same phase). The phase is of the form $\omega t - kz$, where ω is the *angular* frequency (i.e. radians per second) and k the wave vector (i.e. radians per meter). By inspection, $\omega = 10^7\pi$, and since $\omega = 2\pi f$, the frequency is $\boxed{f = \omega/(2\pi) = 5 \times 10^6[\mathrm{Hz}] = 5[\mathrm{MHz}]}$, assuming time is given in seconds.

2.3.2 Part b

FIND

Wavelength.

WORK

The wave number, k, can also be identified by inspection. It is 0.2 (the wave *vector* is $\vec{k} = \hat{z}0.2$, since the wave is forward-propagating in the $\hat{z}$-direction). Since $k = 2\pi/\lambda$, $\boxed{\lambda = 2\pi/k = 2\pi/0.2 = 10\pi \approx 31.42[\mathrm{m}]}$.

There is a correspondence with the *spatial period*, or wavelength, λ, and the temporal period, T: $\omega = 2\pi/T$ and $k = 2\pi/\lambda$.

2.3.3 Part c

FIND

Velocity of light in the medium.

WORK

The phase velocity if this wave is given by $c = \omega/k$. This is $\boxed{c = \dfrac{\omega}{k} = \dfrac{10^7\pi}{0.2} = \dfrac{\pi}{2}10^8 \approx 1.571 \times 10^8[\text{m/s}]}$. For comparison, the speed of light in vacuum is $\approx 3 \times 10^8$ [m/s] .

2.3.4 Part d

FIND

$E(x, y, z, t)$

WORK

E and H can be related through the characteristic impedance, η. The amplitudes follow the relationship, $E = \eta H$, while the cross-product of $\vec{E}$ and $\vec{H}$ points in the direction of the wave vector, $\vec{k}$.

The characteristic impedance, *eta*, is given by $\eta = \sqrt{\frac{\mu}{\epsilon}}$, while the velocity is $c = \frac{1}{\sqrt{\mu\epsilon}}$, so $\eta = \mu c$. If the medium is non-magnetic, then $\mu \approx \mu_0 = 4\pi \times 10^{-7}[\text{H/m}]$,

As such,
$\boxed{\vec{E} = \hat{x}\eta\sin(10^7\pi t - 0.2z) - \hat{y}\eta\sin(10^7\pi t - 0.2z)}$, where

$$\eta = 4\pi \times 10^{-7} \times \frac{\pi}{2}10^8 \tag{2.5}$$
$$= 2\pi^2 \times 10^1 \approx 197.4[\Omega]. \tag{2.6}$$

Again, for comparison, the characteristic impedance of vacuum is approximately 377Ω.

2.3.5 Part e

FIND

Polarization of wave.

WORK

In light of the fact that the wave may be written as (see parts (a) and (d))
$\vec{E} = \sin\left(10^7\pi t - 0.2z\right)\left(\hat{x} - \hat{y}\right),$

we see the wave is *linearly polarized*, and is oriented at an angle of inclination from the $\hat{x}$-axis of -45°.

2.3.6 Part f

FIND

Shortest nonzero time delay, τ, that could be added to the $\hat{x}$-component of the wave in order to convert it to linear polarization? What is then the angle of inclination, θ, of the electric field tip trajectory from the $\hat{x}$-axis?

WORK

For the wave to be linearly-polarized, the $\hat{x}$- and $\hat{y}$-components must be multiples of 180°out of phase with each other (including 0°). This means that the $\hat{x}$- and $\hat{y}$-components must be the same except for amplitude and sign (these may or may not be the same). In this case, we require to include a time delay, τ, such that $\sin(10^7\pi(t-\tau) - 0.2z) = \sin(10^7\pi t - 0.2z)$

Let $A \equiv \omega t - kz$ and $\phi \equiv \omega\tau$, where $\omega = 10^7\pi$ [rad/s] and $k = 0.2$ [rad/m]. Then we require $\sin(A - \phi) = sin(A)$. The minimal value of ϕ for which this is true is zero, since the wave is already linearly polarized. As such, the shortest time delay that could be added is $\tau = 0$, and again, the angle of inclination, θ, is $\theta = -45°$. This is not a big surprise, but it's useful to go through the formalism to see how to turn the crank for these problems.

The required condition will be satisfied for all $\phi = n\pi$, where n is any integer. Then $\tau = \frac{n\pi}{\omega}$. Therefore, the shortest *nonzero* time delay that could be added to the field is $\boxed{\tau = \pi/\omega = 10^{-7} \text{ [s]}}$ (for $n = 1$). In this case, the angle of inclination is shifted by 90°, so that $\boxed{\theta = 45°}$.

2.3.7 Part g

GIVEN

Electric field phasor, $\tilde{\vec{E}} = \hat{x}(j-1) + \hat{y}(1-j)$.

FIND

Polarization.

WORK

By inspection, we see that the magnitudes of the two componets are equal, while the phase angles differ by 180°. Given the discussion in part (f), we know that this is a *linearly-polarized wave.*

Now, let's work out the details. The complex number, $j-1$, is the same as $\sqrt{2}e^{j3\pi/4}$, while $1-j = \sqrt{2}e^{-j\pi/4} = \sqrt{2}e^{j(3\pi/4+\pi)}$. The electric field at $z=0$ is then $\vec{E} = \sqrt{2}\left(\hat{x}\cos\left(\omega t + \frac{3\pi}{4}\right) + \hat{y}\cos\left(\omega t - \frac{\pi}{4}\right)\right) = \sqrt{2}\cos\left(\omega t + \frac{3\pi}{4}\right)(\hat{x} - \hat{y})$, a linearly-polarized wave with angle of inclination, $-45°$.

2.4 Problem 4

GIVEN

1-GHz uniform plane wave propagating in the $\hat{z}$-direction in a medium with permeability, μ, and permittivity, ϵ, characterized by the phasor, $\tilde{\vec{E}} = \hat{x}3$.

2.4.1 Part a

FIND

Time average intensity of the wave.

WORK

For this plane wave, the time average intensity is given by $\boxed{\dfrac{\tilde{E}^2}{2\eta} = \dfrac{9}{2\eta}}$, where $\boxed{\eta \equiv \sqrt{\dfrac{\mu}{\epsilon}}}$.

2.4.2 Part b

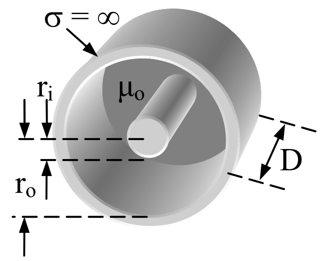
FIND

(i) magnetic energy density, $W_m(t)$, and (ii) electric energy density, $W_e(t)$, at $x = y = z = 0$.

WORK

The magnetic energy density is given from $W_m(t) = \frac{1}{2}\mu \left|\vec{H}\right|^2$, while the electric energy density is given by $W_e(t) = \frac{1}{2}\epsilon \left|\vec{E}\right|^2$. These two quantities are equal, and are both $\boxed{W_m(t) = W_e(t) = \dfrac{1}{2}\epsilon E_0^2 \cos^2(\omega t)}$, where $E_0 = 3$ is the amplitude of the wave, as defined in the phasor. Note that the squared sinusoid produces a DC-component to the energy density as well as a sinusoidal component varying at *twice* the driving frequency, ω.

Problem 5

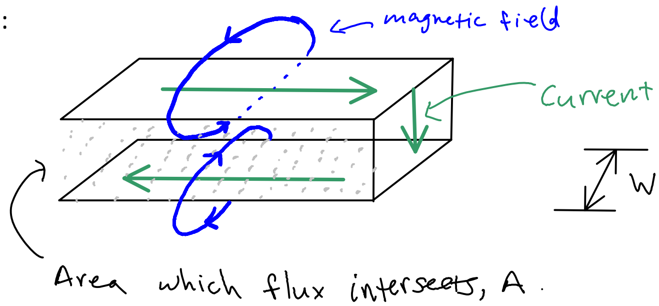


GIVEN: shorted cylindrical coaxial conductor,

FIND: Inductance from general expression,

$$L = \frac{\Lambda}{i} = \frac{\mu N \iint_A \bar{H} \bullet d\bar{a}}{i}$$

WORK: Picture the simple shorted pair of parallel plates for which we calculated our first inductance. It looked like this:



The inductance for this ensemble was $L = \mu A/W$, where A is the area looking into the page, and W is the depth into the page. The larger the loop-enclosed area through which flux cross, the larger the inductance. The further the flux lines travel through the structure, the greater the magnetic reluctance, the weaker the fields, and the smaller the inductance.

In your mind's eye, stretch this shape in the depth direction and wrap it around itself; you will then get the configuration given in this problem. The details of the integration are a little more tedious, but the "meat and potatoes" are the same.

Before getting into the nitty-gritty, note that the shorted end doesn't affect the magnetic fields in either configuration; this is because the field lines are tangential to the surface. Because the enclosure is a perfect conductor, and because $H_{1t} = H_{2t} + J_s$, the shorted end can provide any J_s to support the external tangential field (while preventing the fields from penetrating into the perfect conductor such that H_{2t}=0).

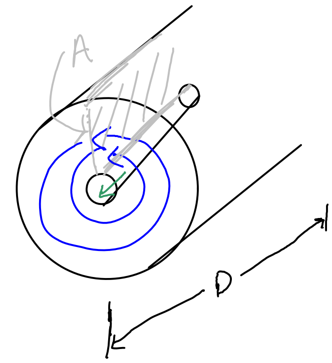
Now, we will calculate the inductance. The program will be to first discern the magnetic field, and then employ the field quantity in the given integral for the inductance. From Ampère's law in DC,

$$\nabla \times \vec{H} = \vec{J} \quad \longleftrightarrow \quad \oint \vec{H} \cdot d\vec{l} = I_{enclosed} = i$$

Circular (azimuthal) symmetry implies that

$$\oint \vec{H} \cdot d\vec{l} = H \oint dl = H\, 2\pi r = i \Rightarrow H = \frac{i}{2\pi r}, \quad \vec{H} = \hat{\theta}\frac{i}{2\pi r}$$

where r is the radial coordinate from the central conductor. Next,



$$L = \frac{\mu N}{i} \iint \vec{H} \cdot d\vec{A}$$

$$(N=1) \qquad = \frac{\mu}{i} \int_0^L \int_{r_i}^{r_o} \frac{i}{2\pi r}\, dr\, dz$$

$$= \frac{\mu}{2\pi} \int_0^L \ln\left(\frac{r_o}{r_i}\right) dz = \boxed{\frac{\mu D \ln\left(r_o/r_i\right)}{2\pi} = L}$$

Let's compare with the square geometry. The bigger the depth, D, the larger the area over which flux crosses, and so the bigger the inductance, as before. But now, the other dimension of this area is linked to the path-length for the field lines, so that the growth in L is much slower (logarithmic) with an increase in this dimension (r_o relative to r_i).