MASSACHUSETTS INSTITUTE OF TECHNOLOGY
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6.007 – Applied Electromagnetics
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Sign Conventions in Electromagnetic (EM) Waves
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Oh, the signs! You may have already encountered some confusion regarding the signs in your homework regarding dissipation of electromagnetic waves. There were several instances of sign confusion in the problem sets and lecture notes, so we apologize for the confusion and will attempt to clarify the issue.

There is actually a difference between physics texts (dealing with general EM, lasers, and all sorts of atomic radiation interactions) and electrical engineering (EE) texts (mostly dealing with antenna theory, where electrical engineers have historically been most prevalent in encountering radiation) in the various definitions of complex dielectric function, complex index of refraction, etc.

For example, in physics texts, you'll see the complex index of refraction listed as:

$$\tilde{n} = n + i\kappa$$

However, in electrical engineering texts (such as the 6.013 course notes), you'll see the index of refraction listed as:

$$\tilde{n} = n - j\kappa$$

Evidently, from these two treatments, it seems that $i = -j$. This is somewhat true in a bizarre way, at least for EM phasor terminology, which we'll elucidate below. **BUT**, this is not a hard and fast convention. Some texts may confuse the two uses, using i and j interchangeably, so we'll show you how to consistently handle signs.

The confusion comes from the initial choice of phasor rotation direction with increasing time, which ultimately decides all the signs in the system.

Choice of phasor rotation direction

If we have a wave defined by:

$$E_y = E_0 \cos(\omega t - \beta z) \tag{1}$$

we can also define this wave using Euler's Theorem:

$$E_y = \frac{E_0}{2} \left(e^{i(\omega t - \beta z)} + e^{-i(\omega t - \beta z)} \right) \tag{2}$$

You'll sometimes see equation (2) written as:

$$E_y = \frac{E_0}{2} \left(e^{i(\omega t - \beta z)} + c.c. \right)$$

or

$$E_y = \frac{E_0}{2} \left(e^{-i(\omega t - \beta z)} + c.c. \right)$$

where the "c.c." stands for "complex conjugate."

Equations (1) and (2) are equivalent and will have only real parts when evaluated. In **linear** systems, we generally don't carry around the extra baggage of the complex conjugate, since we can equivalently just represent E_y by one of the two terms from equation (2) and take the real part, as we've done in class, giving E_y represented by:

$$E_y = Re\left(E_0 e^{i(\omega t - \beta z)} \right) \tag{3}$$

$$\text{or}$$

$$E_y = Re\left(E_0 e^{-i(\omega t - \beta z)}\right) \tag{4}$$

where "Re" means taking the real part of the complex value in the parenthesis.

This is where the confusion about the signs comes in. Do we use equation (3) or (4)?

Mathematically, the difference between equations (3) and (4) is the direction of rotation of the phasor in the complex plane with increasing time. Figure 1 shows this for a phasor $\tilde{x}$.
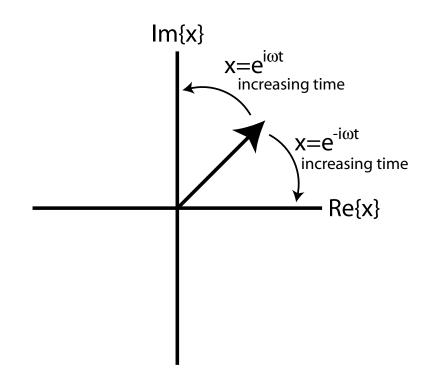

Figure 1: Graphical representation of phasor $\tilde{x}$. Depending on the sign of the exponent, increasing time will cause the phasor to rotate clockwise $(-i)$ or counterclockwise $(+i)$.

In electrical engineering, you'll usually see electromagnetic waves represented using equation (3); in physics, equation (4) is preferred.

Effect of choice on index of refraction definition

Going back to equation (2), if we want to add some sort of dissipation which causes the magnitude of the electric field to decrease as the wave travels further in $+\hat{z}$, we represent this mathematically as:

$$E_y = e^{-\alpha z}\frac{E_0}{2}\left(e^{i(\omega t - \beta z)} + e^{-i(\omega t - \beta z)}\right) \tag{5}$$

where α has units of inverse distance, or $\text{m}^{-1} = 1/\text{m}$. **We are specifying here that $\alpha > 0$ corresponds to the case of dissipation**.

If we combine terms, placing $-\alpha z$ in the two complex exponentials, we get:

$$
\begin{aligned}
E_y &= \frac{E_0}{2}\left(e^{i(\omega t - \beta z + i\alpha z)} + e^{-i(\omega t - \beta z - i\alpha z)}\right) \\
&= \frac{E_0}{2}\left(e^{i(\omega t - (\beta - i\alpha)z)} + e^{-i(\omega t - (\beta + i\alpha)z)}\right)
\end{aligned}
\tag{6}
$$

Equation (6) shows two different possible definitions for the complex wavevector, $\tilde{\beta}$. If you use the first term (EE definition) in equation (6), then $\tilde{\beta}$ is given by:

$$
\tilde{\beta} = \beta - i\alpha
\tag{7}
$$

If you use the second term (physics definition) in equation (6), you get:

$$
\tilde{\beta} = \beta + i\alpha
\tag{8}
$$

This then leads straightforwardly to the two different definitions for the index of refraction since $\tilde{\beta} = \tilde{n}\frac{\omega}{c}$, or $\tilde{n} = \frac{c}{\omega}\tilde{\beta}$.

Using the EE definition:

$$
\begin{aligned}
\tilde{n} &= \frac{c}{\omega}\left(\beta - i\alpha\right) \\
&= n - i\kappa
\end{aligned}
\tag{9}
$$

Using the physics definition:

$$
\begin{aligned}
\tilde{n} &= \frac{c}{\omega}\left(\beta + i\alpha\right) \\
&= n + i\kappa
\end{aligned}
\tag{10}
$$

Since we typically use j instead of i in electrical engineering, you typically see equation (9) written as:

$$
\begin{aligned}
\tilde{n} &= \frac{c}{\omega}\left(\beta - j\alpha\right) \\
&= n - j\kappa
\end{aligned}
$$

hence resulting in the "equivalence" of $i = -j$, which really only means that physicists and electrical engineers use different phasor notation for their EM waves.

Some other effects of the choice of phasor representation

One mistake in Problem Set 6 that you may have noticed when trying to solve problem #4 was that using the given Lorentz oscillator model equation, the wave in silicon did not dissipate but rather was amplified as it traveled to larger z.

This was due to an incorrect sign in the Lorentz oscillator model equation given. The one in the problem set was:

$$
\epsilon_r(\omega) = 1 + \frac{\omega_p^2}{\omega_0^2 - \omega^2 - i\omega\gamma}
$$

This version of the Lorentz model equation is derived by representing the polarization in the medium as:

$$
P(z,t) = P(\omega, z)e^{-i\omega t}
$$

which, when differentiated in the derivation, gives a factor $-i\omega$.

The use of $-i\omega t$, which is from the physics phasor notation, means that we inadvertently combined the physics and electrical engineering notations, which will result in sign errors.

The Lorentz model equation we should have used would have been derived using:

$$P(z,t) = P(\omega, z)e^{j\omega t}$$

which would have then given us a version of the Lorentz model consistent with the EE phasor representation of the E-field of the electromagnetic wave:

$$\epsilon_r(\omega) = 1 + \frac{\omega_p^2}{\omega_0^2 - \omega^2 + j\omega\gamma}$$

Summary: Which notation to use? Just be consistent!

In 6.007, we'll be using the EE notation.

Ultimately, your choice of notation does not matter as long as you are consistent. This notation difference between physics and EE, along with the different definition of right and left-handed polarization between the two fields, leads to many headaches when trying to learn the material, but it's an unfortunate historical reality.

For what we've covered in 6.007, below are the consistent representations for EE and physics notation:

Equation	Electrical Engineering	Physics
Electric Field	$E_y = Re\left(\tilde{E}_y e^{j(\omega t - \tilde{\beta}z)}\right)$	$E_y = Re\left(\tilde{E}_y e^{-i(\omega t - \tilde{\beta}z)}\right)$
Wavevector, $\tilde{\beta}$	$\tilde{\beta} = \beta - j\alpha$	$\tilde{\beta} = \beta + i\alpha$
Index of Refraction, $\tilde{n}$	$\tilde{n} = n - j\kappa$	$\tilde{n} = n + i\kappa$
Lorentz Model, $\tilde{\epsilon}_r$	$\tilde{\epsilon}_r = 1 + \frac{\omega_p^2}{\omega_0^2 - \omega^2 + j\omega\gamma}$	$\tilde{\epsilon}_r = 1 + \frac{\omega_p^2}{\omega_0^2 - \omega^2 - i\omega\gamma}$

The electric field wave shown above is for general cases. The phasor $\tilde{E}_y$ has a magnitude and can have a non-zero phase. A non-zero phase would correspond to cases where the E-field goes not as a pure cosine, but rather as a sine or a mixture of sine and cosine.

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