

6.007 – Applied Electromagnetics
Spring 2011

Problem Set 7

Due Wednesday April 6, 2011

Problem 7.1, Atmospheric CO₂ Measurement

Working at the Mauna Loa Observatory in beautiful, sunny Hawaii, you perform infrared absorption spectroscopy to measure the concentration of CO₂ in air samples from the upper atmosphere. In their own words—

[...] Air is slowly pumped through a small cylindrical cell with flat windows on both ends. Infrared light is transmitted through one window, through the cell, through the second window, and is measured by a detector that is sensitive to infrared radiation. In the atmosphere carbon dioxide absorbs infrared radiation, contributing to warming of the earth surface. Also in the cell CO₂ absorbs infrared light. More CO₂ in the cell causes more absorption, leaving less light to hit the detector. We turn the detector signal, which is registered in volts, into a measure of the amount of CO₂ in the cell through extensive and automated (always ongoing) calibration procedures.¹

In this problem, you will use the Lorentz oscillator model to predict the peak absorption of atmospheric CO₂ by estimating its plasma frequency ω_p , resonance frequency ω_0 , and linewidth γ .

1. For a dilute gas (i.e., a medium with small susceptibility), show that,

$$n = \sqrt{\epsilon_r} \quad \kappa = \frac{\epsilon_i}{2n}. \quad (1)$$

2. Use the most recent (mean) CO₂ concentration (300 ppm) measured at Mauna Loa and the Ideal Gas Law to determine the number of CO₂ molecules per unit volume m³. Assume temperature is 300 °K. Estimate the plasma frequency of this sample. What is an approximate index of refraction for a sample of this gas?
3. Figure 1 shows the absorption spectrum of CO₂ (taken from Lecture 20). Most infrared CO₂ lasers

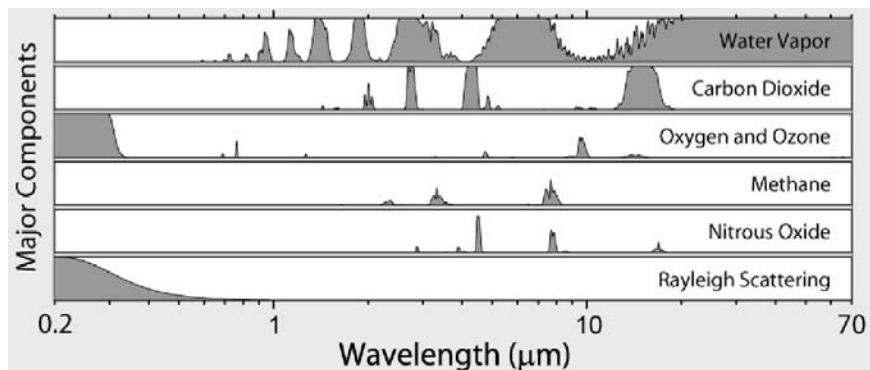


Figure 1: Observed gas absorption spectrum in the upper-atmosphere.

Image created by Robert A. Rohde / Global Warming Art. Used with permission.
http://www.globalwarmingart.com/wiki/File:Atmospheric_Absorption_Bands_png.

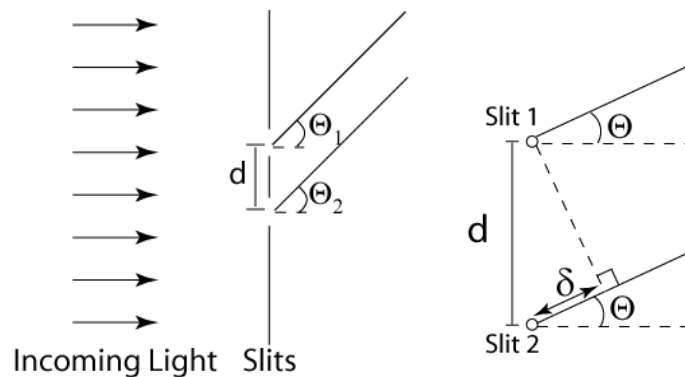
emit at 10.6 μm. From this graph, determine the center frequency ω_0 and linewidth γ (measured full width at half-maximum) of the CO₂ absorption peak.

¹Mauna Loa has written about their measure procedure and mean measurements (since 1958!) at http://www.esrl.noaa.gov/gmd/ccgg/trends/co2_data_mlo.html.

4. What is the *peak* absorption coefficient at this absorption line? Compare this to the absorption line measured for 350ppm.
5. If a 1mW beam at $10.6\mu\text{m}$ wavelength hits CO_2 , what is the output power after traveling 10m?
6. Will the absorption coefficient measured in an experiment be smaller or larger than the calculated value? Why?

Problem 7.2, Two Slit Spectrometer

Following the results of the previous problem, you would like to make a measurement of the concentration of the carbon dioxide in the air. To do this we would like to look at the absorption of the air across the range of wavelengths carbon dioxide has some absorption. You could try measuring the absorption at individual wavelengths with discrete detectors and sources, but it would be faster if you could make one measurement at several wavelengths at the same time. To make the measurement in one shot you use something called a spectrometer. In this problem we will look at a spectrometer based on the two slit grating discussed in class.



- (a) Using theory from the two slit interference pattern shown in class, find the angle of the first off axis interference maximum (first order fringe) as a function of slit distance and wavelength. Assume the width of the slits is negligible.
- (b) If we are interested in the wavelength region from $9 - 11\mu\text{m}$ how far away from the slits would we need to place our camera so that this wavelength band is spread over 1 cm. Assume that the slits are separated by $d = 50\mu\text{m}$ and that we want the $10\mu\text{m}$ wavelength to occur in the center of the pattern.
- (c) How does the interference pattern change as we increase the number of slits?

To answer this question you can use the applet at:

<http://physics.neiu.edu/vpl/optics/diffraction.html>.

For the applet, set the slit width to 0.1 (small enough that the slits are approximately point sources) and wavelength to 10.

- (d) A CCD camera is placed 1 m away from the slits. The pixel size is 20 microns, what is the wavelength resolution of this CCD+slit spectrometer if the fringes were infinitely sharp? Again assume a slit separation of $d = 50\mu\text{m}$ and that the $10\mu\text{m}$ fringe is at the middle of the CCD.

Problem 7.3

Plane waves from the same source can form an interference pattern (subject to some conditions we'll ignore here). In practice that means that when waves reflect from boundaries a standing wave pattern can be formed on the side of the boundary containing the incident and reflected waves.

The figures on the next page indicate the standing wave patterns resulting when an incident wave in medium 1 with amplitude equal to 1 V/m is incident on an interface. Label the graphs (a)-(d) to match them with the description detailed below. Note that not all figures have a description (10 figures, 4 descriptions).

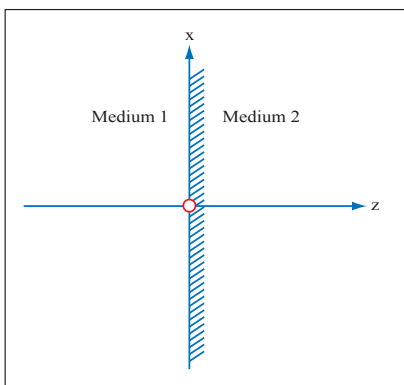


Image by MIT OpenCourseWare.

- (a) Plot of $|E_{y,total}|$ with medium 1 being air, medium 2 $n_2 = 2$. Normal Incidence
- (b) Plot of $|E_{y,total}|$ with medium 1 $n_1 = 2$, medium 2 being air. Normal Incidence
- (c) Plot of $|E_{y,total}|$ with medium 1 being air, medium 2 being a perfect conductor
- (d) Plot of $|H_{y,total}|$ with medium 1 being air, medium 2 being a perfect conductor

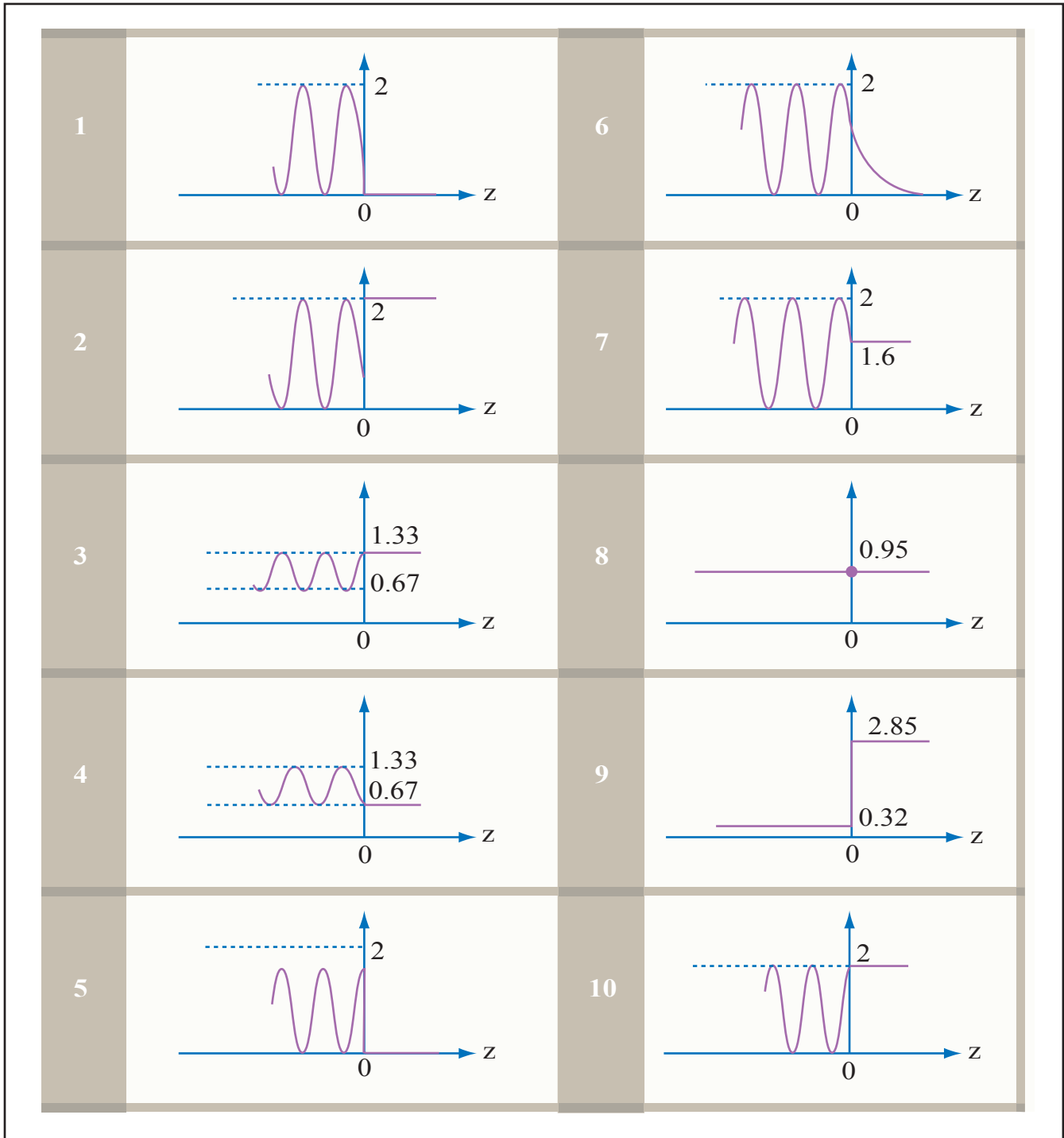
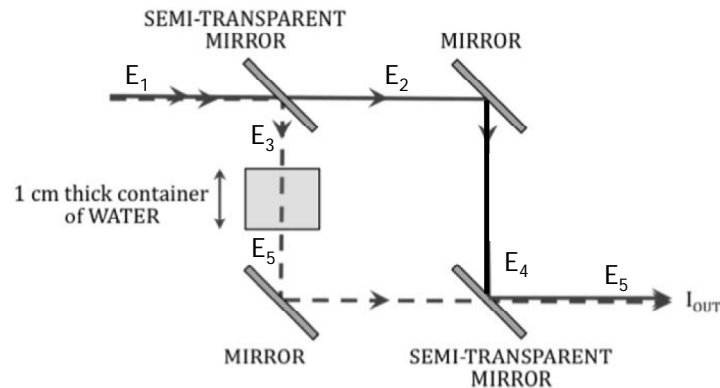


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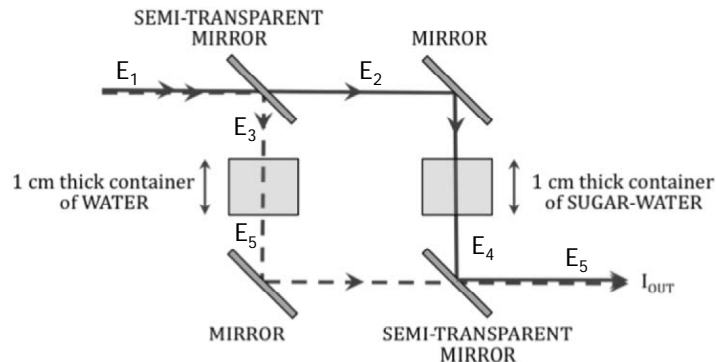
Problem 7.4 Mach-Zehnder Interferometer

Mach and Zehnder propose to use an interferometer to measure how much sugar is in a SUGAR-WATER sample as compared to a control sample containing only WATER. The basic idea is to split a beam of light into two beams (which are initially in phase), and send each through an identical path length, and then recombine them (see the figure). If the two beams at the output are in phase, then they will constructively interfere, and the signal observed, I_{out} , is maximized. If they destructively interfere, then I_{out} will be minimized.



Assume the refractive index of the air is 1, and the refractive index of the water is 1.33. The refractive index of water is increased by 0.002 for each gram of sugar added to water.

- First assume one beam (solid line) passes through nothing but air, but the second beam (dotted line) passes through a container of a sugar solution in water. Assume that containers are made of optically transparent materials and the thickness of the container is d . Write expressions for the electric fields E_1 through E_5 at the points indicated on the graph. Answers can be left in terms of the index of the solution n_s and the wavevector k . Now find the intensity I_{out} . How does the intensity change as the solution concentration increases?
- Assume the thickness of the container is 1 cm and the wavelength is 633 nm. How much sugar must be added to the solution to make the I_{out} equal to zero? Where do you think the energy goes in this case?
- Now assume you have an identical container of 1 cm thickness in the first beam but it's filled with water. How much sugar must be added to the solution in the container in the sensing arm to minimize I_{out} now? Explain the difference in the sensitivity of the Mach-Zehnder in these two cases.

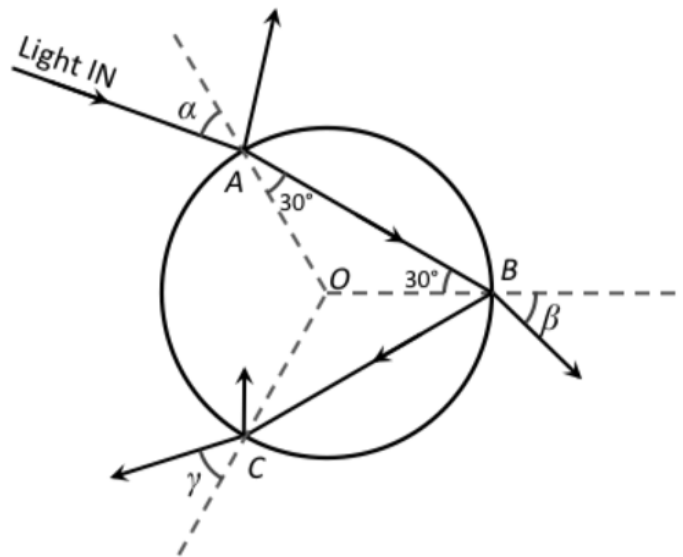


- Consider using this device as a sensor for refractive index, and the only signal you have is an intensity readout from a power meter placed at the output. You first take a calibration with a solution that has

a known amount of sugar dissolved. Then, your friend adds some sugar to the starting solution and you are allowed to measure again the output intensity with this solution placed in the Mach-Zehnder sensor. Assuming the change in index between this and the reference solution is small, what is the concentration that you would choose to start with? This is called biasing the Mach-Zehnder

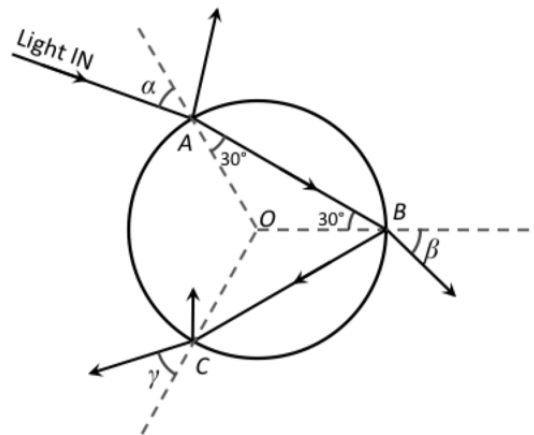
- (e) Consider the same sensing configuration as part (d) but the first semitransparent mirror is not perfect and splits the light beams in a ratio of 40:60 instead of 50:50, with less intensity passing through the container with sugar. What are the expressions now for the E fields and of I_{out} ? With the same biasing condition as before, does this increase or decrease your sensitivity?
- (f) Now you have the choice of using either a light source of 633 nm or 800 nm to perform the intensity measurements described above. Which wavelength would you use to have the best sensitivity?

Problem 7.5 Rainbows



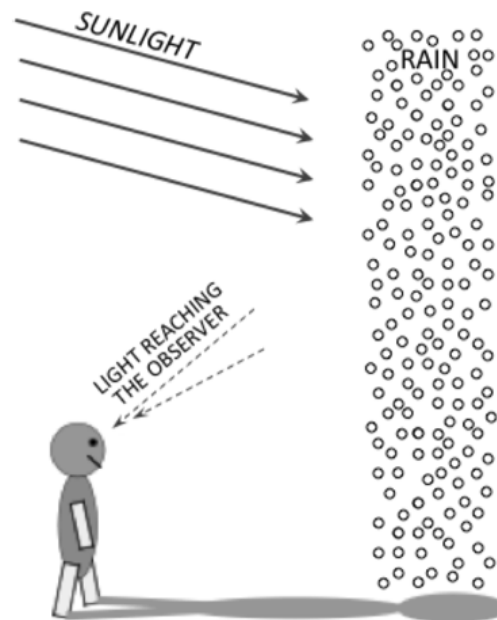
A very narrow beam of unpolarized red light of intensity I_o is incident (at A) on a spherical water drop (see figure above). At A, some of the light is reflected and some enters the water drop. The refracted light reaches the surface of the drop at B where some of the light is reflected back into the water, and some emerges into the air. The light that is reflected back into the water reaches the surface of the drop at C where some of the light is reflected back into the drop, and some emerges into the air. The index of refraction of water for the red light is $n_{red} = 1.331$.

- (a) Using the data from the figure, what is the angle α ? (You can leave this answer in the form of an expression.)
- (b) What is the intensity of light that refracts into the drop at A? (Take into account both TE and TM polarized light.)
- (c) For what value of angle would you find that the light reflected at A is entirely TE polarized? (You can leave this answer in the form of an expression.)
- (d) What is the dominant polarization of light that emerges at B? Explain.



(e) For blue light, the index of refraction in water is 1.343. The speed of blue light in water is therefore about 1% slower than that of red light. In the figure above, showing the red-light path, assume that the incoming narrow beam of light also contains blue-light and draw the trajectory that the blue-light beam would take after it enters the water droplet at A .

(f) If you look at the rainbow in the sky you will notice that the blue band of color is closest to the ground and the red color band is highest up. Using the figure below explain why the colors are ordered in this manner.



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