6.006 Introduction to Algorithms
Spring 2008

6.006 Recitation

Build 2008.14

Coming up next...

- Open addressing
- Karp-Rabin
  - coming back from the dead to hunt us

Open Addressing

- Goal: use nothing but the table
  - Hoping for less code, better caching
- Hashing $\Rightarrow$ we must handle collisions
  - Solution: try another location

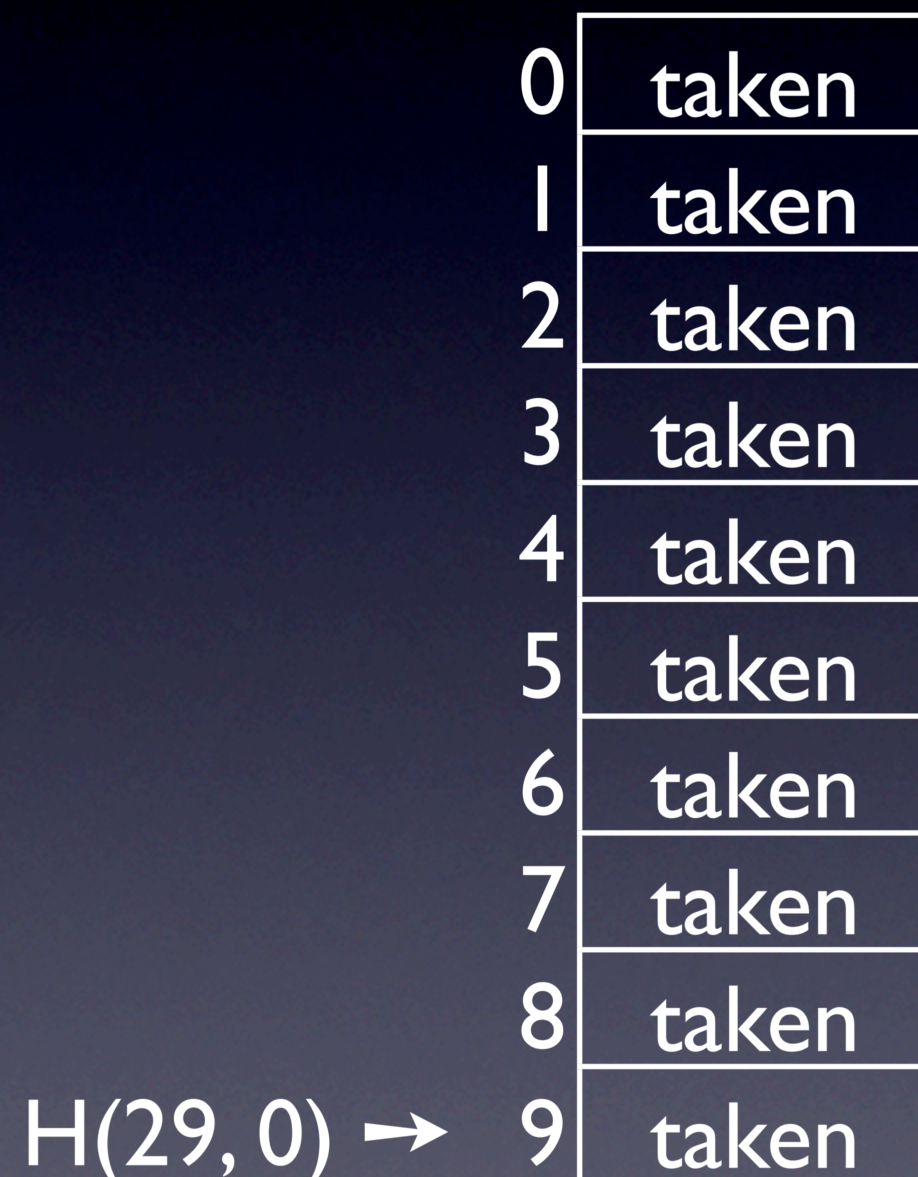
Easy Collision handling

- h(x) = standard hash function
- if T[h(x)] is taken
  - try T[h(x)+1]
  - then T[h(x) + 2]
  - then T[h(x) + 3]
- just like parking a car

	0	taken
	1	
	2	taken
	3	
h(29) →	4	taken
h(29) + 1 →	5	taken
h(29) + 2 →	6	taken
h(29) + 3 →	7	here ☺
	8	
	9	taken

Collision Handling: Abstracting it Up

- h(k) grows up to H(k, i) where i is the attempt number
- first try T[H(k, 0)]



Collision Handling: Abstracting it Up

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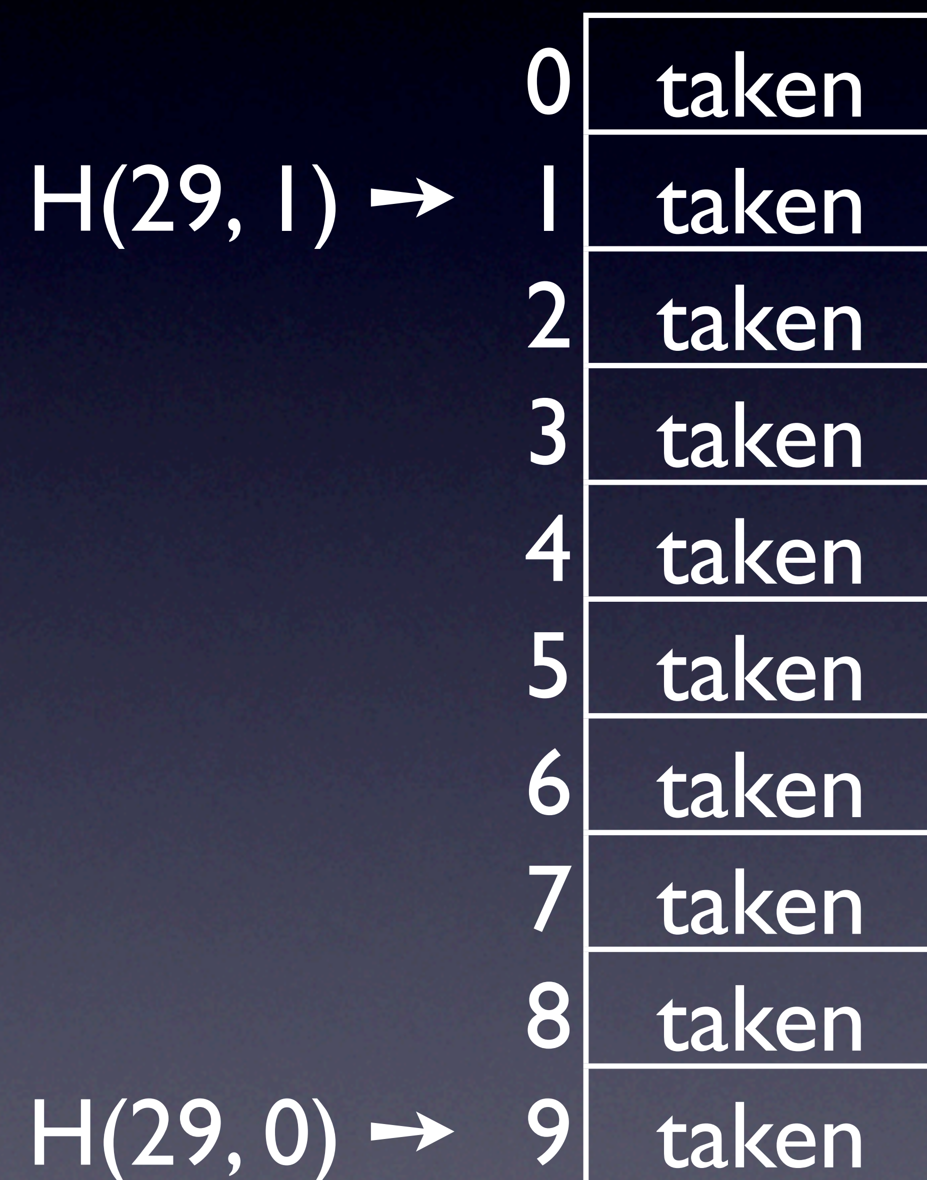

Collision Handling: Abstracting it Up

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- first try T[H(k, 0)]
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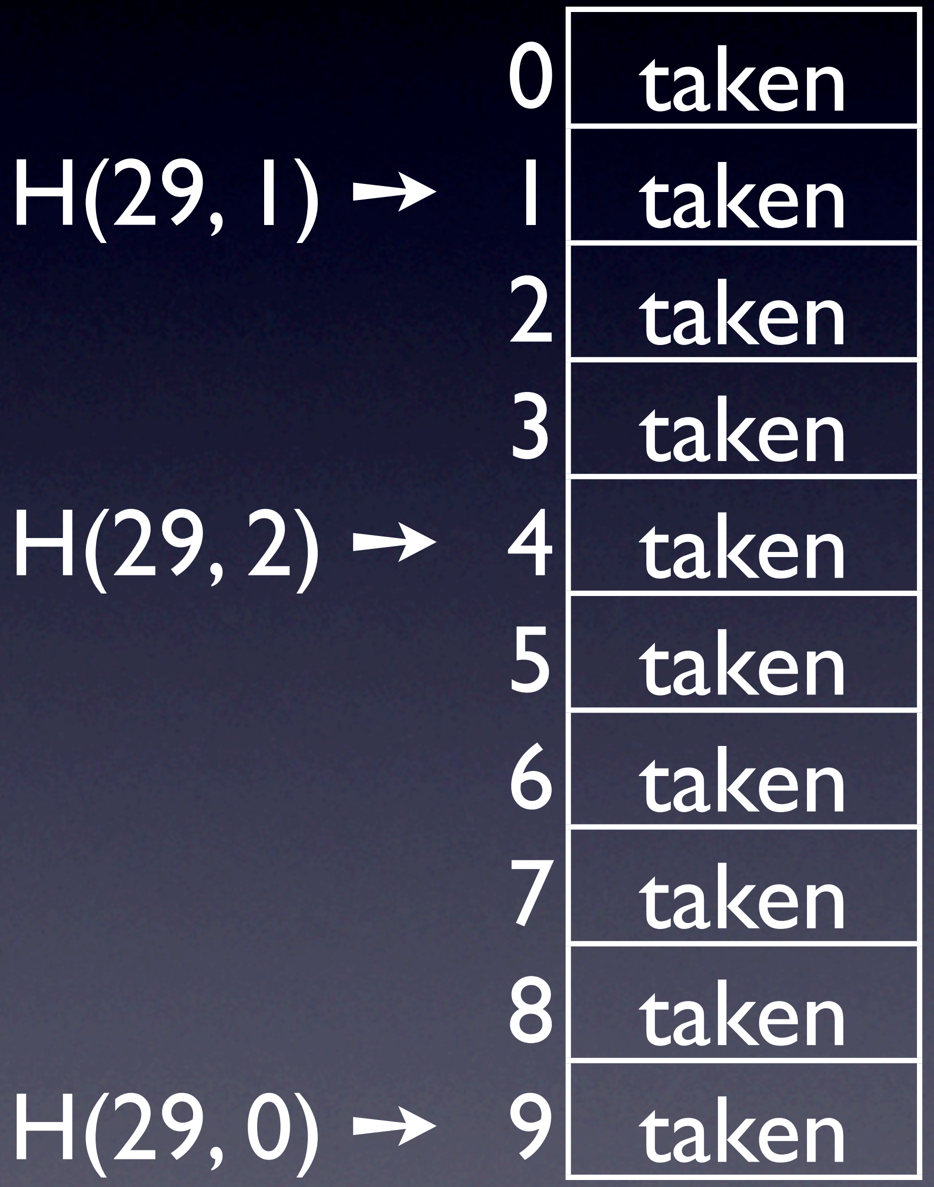


Collision Handling: Abstracting it Up

- h(k) grows up to H(k, i) where i is the attempt number
- first try T[H(k, 0)]
  - then T[H(k, 1)]
  - then T[H(k, 2)]
- stop after trying all

H(29, 3)	→	0	taken
H(29, 1)	→	1	taken
H(29, 4)	→	2	taken
H(29, 9)	→	3	taken
H(29, 2)	→	4	taken
H(29, 5)	→	5	taken
H(29, 6)	→	6	taken
H(29, 7)	→	7	taken
H(29, 8)	→	8	taken
H(29, 0)	→	9	taken

Collision Handling: Abstracting it Up

- H(k) = <H(k, 0), H(k, 1), H(k, 2) ... >
- Linear probing, h(29) = 4, $H_{linear}(29)$ = ?

  <4, 5, 6, 7, 8, 9, 0, 1, 2, 3>

- General properties?

H(29, 3) →	0	taken
H(29, 1) →	1	taken
H(29, 4) →	2	taken
H(29, 9) →	3	taken
H(29, 2) →	4	taken
H(29, 5) →	5	taken
H(29, 6) →	6	taken
H(29, 7) →	7	taken
H(29, 8) →	8	taken
H(29, 0) →	9	taken

Collision Handling: Abstracting it Up

- Any collision handling strategy comes to:
  - for key k, probe H(k,0), then H(k,1) etc.
- No point in trying the same place twice
- Probes should cover the whole table (otherwise we raise 'table full' prematurely)
- Conclusion: H(k, 0), H(k, 1) ... H(k, m-1) are a permutation of {1, 2, 3 ... m}

Linear Probing and Permutations

- $h(29) = 4$; $H(29) =$

  $\langle 4, 5, 6, 7, 8, 9, 0, 1, 2, 3 \rangle$

- $h(k) = h_0 (\text{mod } m)$; $H(k) =$

  $\langle h_0 \text{ mod } m, (h_0 + 1) \text{ mod } m, (h_0 + 2) \text{ mod } m, \ldots (h_0 + m - 1) \text{ mod } m \rangle$

- m permutations (max m!)

	0	taken
	1	
	2	taken
	3	
h(29) →	4	taken
h(29) + 1 →	5	taken
h(29) + 2 →	6	taken
h(29) + 3 →	7	here ☺
	8	
	9	taken

Ideal Collision Handling

- Simple Hashing (collision by chaining)
  - Ideal hashing function: uniformly distributes keys across hash values
- Open Addressing
  - Ideal hashing function: uniformly distributes keys across permutations
  - a.k.a. uniform hashing

Uniform Hashing: Achievable?

- Simple mapping between permutations of m and numbers 1 ... m!
- Convert key to big number, then use permutation number (bignum mod m!)
- ... right?

k mod 6	Permutation
0	<1, 2, 3>
1	<1, 3, 2>
2	<2, 1, 3>
3	<2, 3, 1>
4	<3, 1, 2>
5	<3, 2, 1>

Uniform Hashing: Achievable?

- Number of digits in m!
    - O(log(m!))
    - O(m*log(m))
- Working mod m! is slow
    - check your Python cost model

k mod 6	Permutation
0	<1, 2, 3>
1	<1, 3, 2>
2	<2, 1, 3>
3	<2, 3, 1>
4	<3, 1, 2>
5	<3, 2, 1>

Working Compromise

- Why does linear probing suck?
  - We jump in the table once, then walk
- Improvement
  - Keep jumping after the initial jump
  - Jumping distance: 2nd hash function
  - Name: double hashing

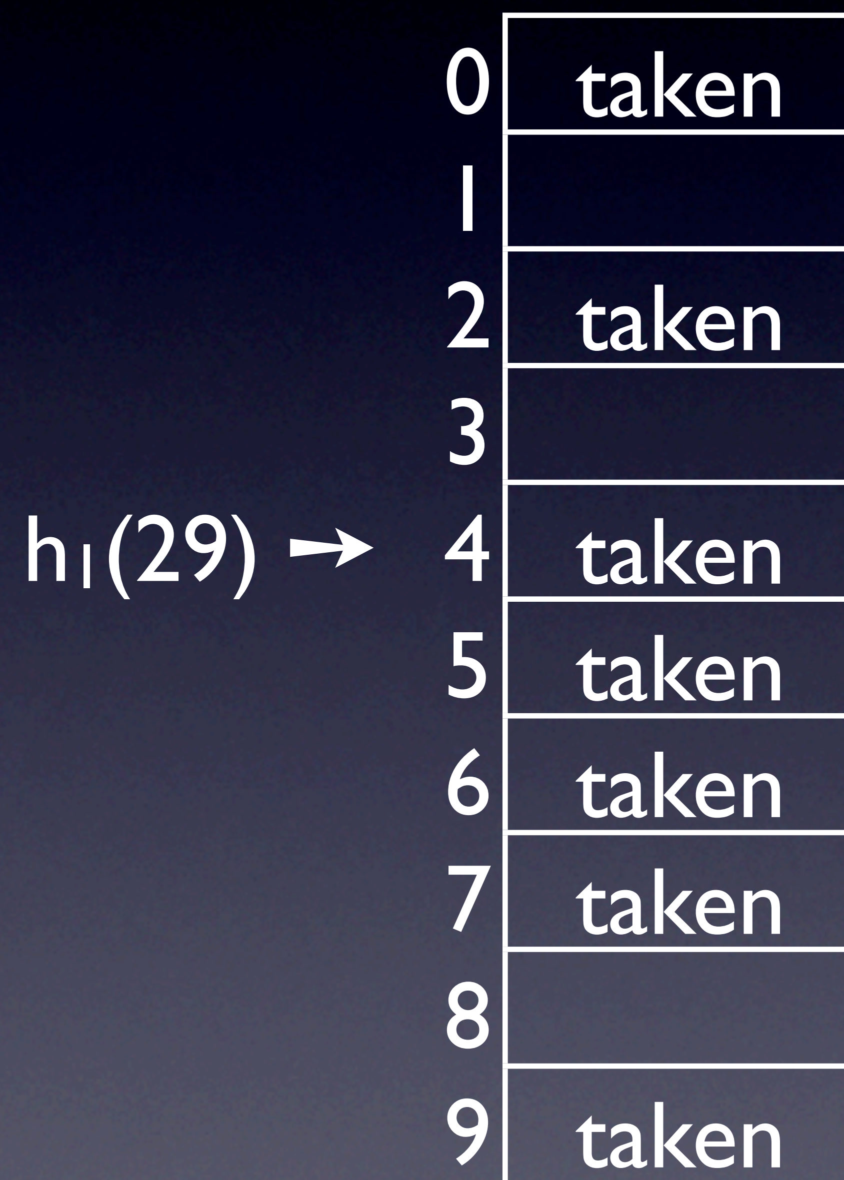
Double Hashing: Math

- $h_1(k)$ and $h_2(k)$ are hashing functions

0	taken
1	
2	taken
3	
4	taken
5	taken
6	taken
7	taken
8	
9	taken

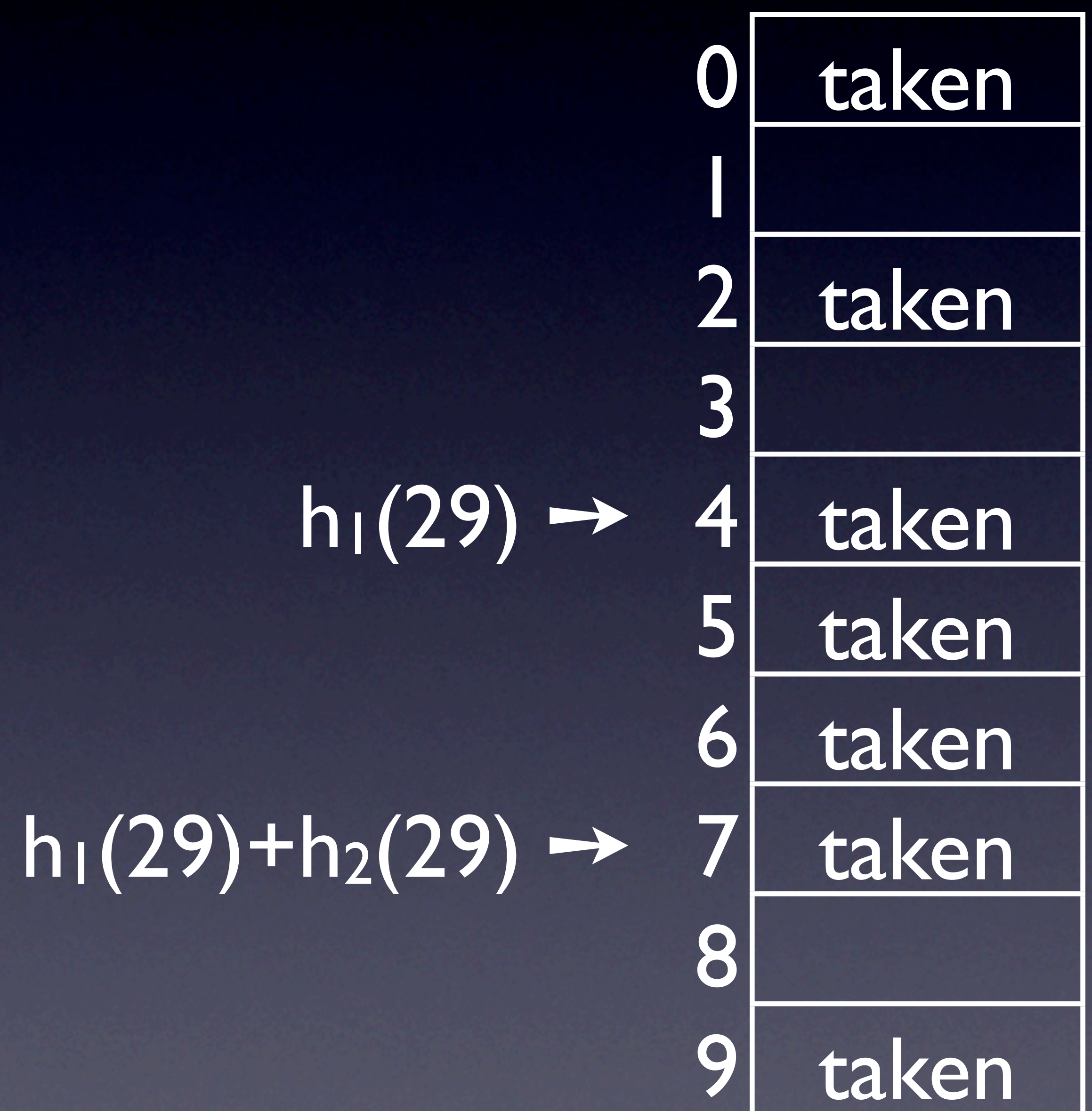
Double Hashing: Math

- $h_1(k)$ and $h_2(k)$ are hashing functions
- $H(k, 0) = h_1(k)$



Double Hashing: Math

- $h_1(k)$ and $h_2(k)$ are hashing functions
- $H(k, 0) = h_1(k)$
- $H(k, 1) = h_1(k) + h_2(k)$



Double Hashing: Math

- $h_1(k)$ and $h_2(k)$ are hashing functions
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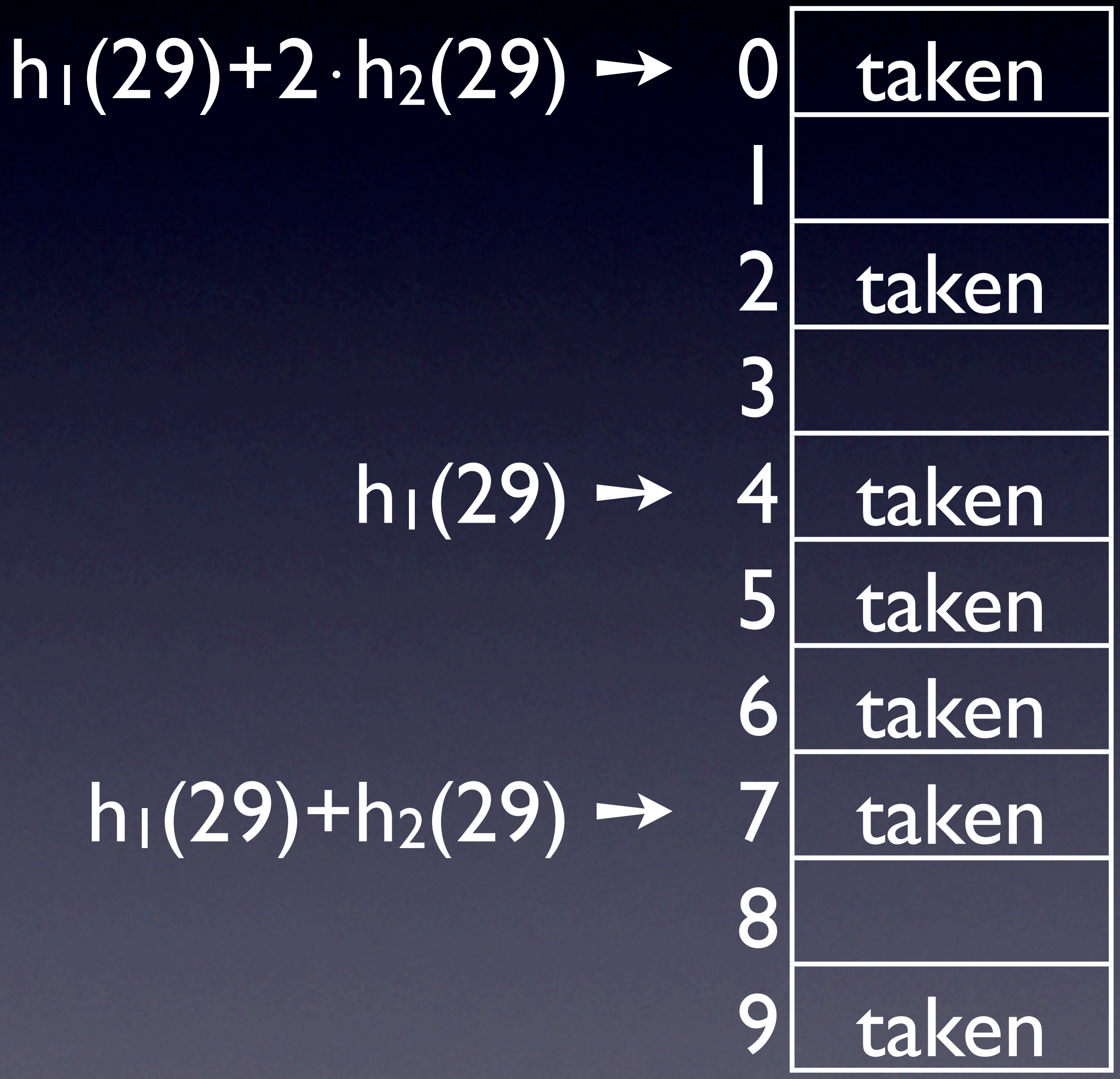


Double Hashing: Math

- $h_1(k)$ and $h_2(k)$ are hashing functions
- $H(k, 0) = h_1(k)$
- $H(k, 1) = h_1(k) + h_2(k)$

$h_1(29)+2 \cdot h_2(29)$ →	0	taken
	1	
	2	taken
$h_1(29)+3 \cdot h_2(29)$ →	3	here ☺
$h_1(29)$ →	4	taken
	5	taken
	6	taken
$h_1(29)+h_2(29)$ →	7	taken
	8	
	9	taken

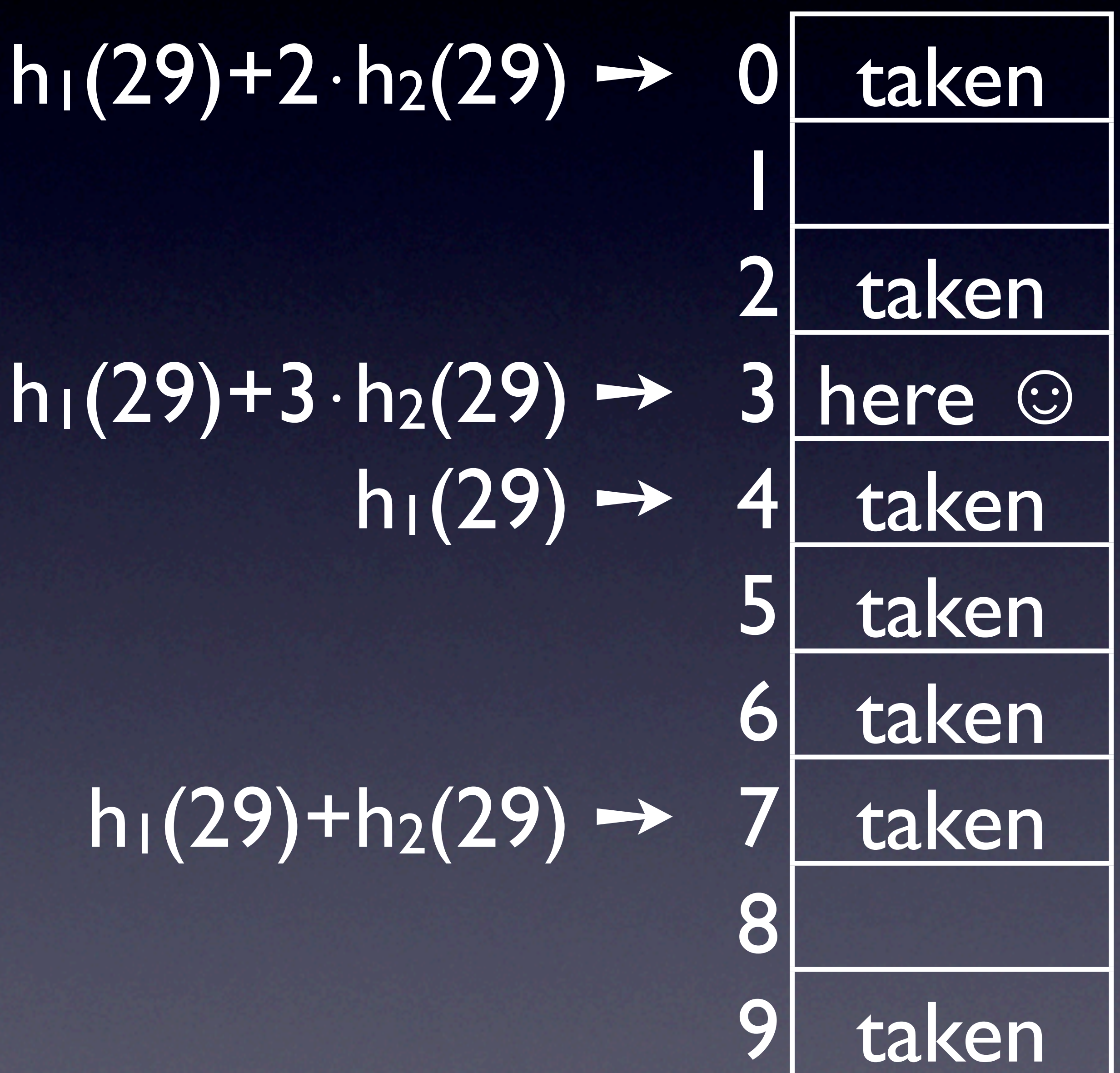
Double Hashing: Math

- $h_1(k)$ and $h_2(k)$ are hashing functions
- $H(k, 0) = h_1(k)$
- $H(k, 1) = h_1(k) + h_2(k)$
- $H(k, i) = h_1(k) + i \cdot h_2(k)$
  - mod m
  - you knew that, right?

$h_1(29)+2 \cdot h_2(29)$ →	0	taken
	1	
	2	taken
$h_1(29)+3 \cdot h_2(29)$ →	3	here ☺
$h_1(29)$ →	4	taken
	5	taken
	6	taken
$h_1(29)+h_2(29)$ →	7	taken
	8	
	9	taken

Double Hashing Trap

- gcd($h_2(k)$, m) must be 1
- solution 1 (easy to get)
  - m prime, $h_2(k) = k \bmod q$ (where $q < m$)
- solution 2 (faster, better)
  - $m = 2^r$ (table can grow)
  - $h_2(k)$ is odd (not even)



Open Addressing: Deleting Keys

- Suppose we want to delete k_d stored at 7
- Can't simply wipe the entry, because key 29 wouldn't be found anymore
  - rember H(29) = <4, 7, 0, 3 ...>

$h_1(29)+2 \cdot h_2(29)$ →	0	taken
	1	
	2	taken
$h_1(29)+3 \cdot h_2(29)$ →	3	here ☺
$h_1(29)$ →	4	taken
	5	taken
	6	taken
$h_1(29)+h_2(29)$ →	7	k_d
	8	
	9	taken

Open Addressing: Deleting Keys

- Entry meaning 'deleted'
- Handling 'deleted'
  - Search: Keep looking
  - Insert: Stop, replace 'deleted' with the new key/value

$h_1(29)+2 \cdot h_2(29)$ →	0	taken
	1	
	2	taken
$h_1(29)+3 \cdot h_2(29)$ →	3	here ☺
$h_1(29)$ →	4	taken
	5	taken
	6	taken
$h_1(29)+h_2(29)$ →	7	deleted
	8	
	9	taken

Open Addressing:Code

- Design: implementing a collection in Python
  - **__getitem__**(self, key)
    - return key item or raise KeyError(key)
  - **__setitem__**(self, key, item)
    - insert / replace (key, item)
  - **__delitem__**(self, key)

Open Addressing: Code

- Closures: not for n00bs
- `def compute_modulo` is local to the `mod_m` call
- the function created by `def compute_modulo` is returned like any object
- the object remembers the context around the def (the value of m)

```python
def mod_m(m):
    def compute_modulo(n):
        return (n % m)
    return compute_modulo

>>> m5 = mod_m(5)
>>> m3 = mod_m(3)
>>> m5(9)
4
>>> m3(9)
0
```

Open Addressing:Code

```python
def linear_probing(m = 1009):
    def hf(key, attempt):
        return (hash(key) + attempt) % m
    return hf

def double_hashing(hf2, m = 1009):
    def hf(key, attempt):
        return (hash(key) + attempt * hf2(key)) % m
    return hf

class DeletedEntry:
    pass

class OpenAddressingTable:
    def __init__(self, hash_function, m = 1009):
        self.entries = [None for i in range(m)]
        self.hash = hash_function
        self.deleted_entry = DeletedEntry()
```

Open Addressing: Code

```python
class OpenAddressingTable:
    def __init__(self, hash_function, m = 1009):
        self.entries = [None for i in range(m)]
        self.hash = hash_function
        self.deleted_entry = DeletedEntry()

    def get_entry(self, key):
        for attempt in xrange(len(self.entries)):
            h = self.hash(key, attempt)
            if self.entries[h] is None:
                return None
            if self.entries[h] is not self.deleted_entry and \
               self.entries[h][0] == key:
                return self.entries[h]

    def __getitem__(self, key):
        entry = self.get_entry(key)
        if entry is None:
            raise KeyError(key)
        return entry[1]

    def __contains__(self, key):
        return self.get_entry(key) is not None
```

Open Addressing: Code

```python
class OpenAddressingTable:
    def __init__(self, hash_function, m = 1009):
        self.entries = [None for i in range(m)]
        self.hash = hash_function
        self.deleted_entry = DeletedEntry()

    def __setitem__(self, key, value):
        if value is None: raise 'Cannot set value to None'
        del self[key]
        for attempt in xrange(len(self.entries)):
            h = self.hash(key, attempt)
            if self.entries[h] is None or \
               self.entries[h] is self.deleted_entry:
                self.entries[h] = (key, value)
                return
        raise 'Table full'
```

Open Addressing: Code

```
class OpenAddressingTable:
    def __init__(self, hash_function, m = 1009):
        self.entries = [None for i in range(m)]
        self.hash = hash_function
        self.deleted_entry = DeletedEntry()

    def __delitem__(self, key):
        for attempt in xrange(len(self.entries)):
            h = self.hash(key, attempt)
            if self.entries[h] is None:
                return
            if self.entries[h] is not self.deleted_entry and \
               self.entries[h][0] == key:
                self.entries[h] = self.deleted_entry
                return
        return
```

Ghosts of Karp & Rabin

Getting Rolling Hashes Right

Modular Arithmetic

- Foundation:
  - $(a + b) \bmod m = ((a \bmod m) + (b \bmod m)) \bmod m$
- From that, it follows that:
  - $(a \cdot b) \bmod m = ((a \bmod m) \cdot (b \bmod m)) \bmod m$
    - induction: multiplication is repeated +

Modular Gotcha

- Never give mod a negative number
  - want q = (a - b) mod m, but a - b < 0
  - q mod m = (a - (b mod m)) mod m
  - but (b mod m) is < m
  - so (a + m - (b mod m)) > 0
  - q = (a + m - (b mod m)) mod m

Modular Arithmetic-Fu

- Multiplicative inverses: assume p is prime
- For every a and p, there is a^{-1} so that:
  - $(a * a^{-1}) \bmod p = 1$
  - example: $p = 23$, $a = 8 \Rightarrow a^{-1} = 3$
    - check: $8 * 23 = 24$, $24 \bmod 23 = 1$
- Multiplying by a^{-1} is like dividing by a

Modular Arithmetic-Fu

- How do we compute a^{-1}?
- Fermat's Little Theorem:
  - p prime $\Rightarrow a^{a-1} \bmod p = 1$
- Huh?
  - $a^{a-1} \bmod p = a * a^{a-2} \bmod p = 1$
  - so (for p) $a^{-1} \bmod p = a^{a-2} \bmod p$

Back to Rolling Hashes

- Data Structure (just like hash table)
  - start with empty list
  - append(val): appends val at the end of list
  - skip(): removes the first list element
  - hash(): computes a hash of the list