6.003 Homework #3 Solutions

Problems

1. Complex numbers

a. Evaluate the real and imaginary parts of j^j.

Real part = $e^{-\pi/2}$ Imaginary part = 0

Euler's formula says that $j = e^{j\pi/2}$, so

$$j^j = \left(e^{j\pi/2}\right)^j = e^{-\pi/2}.$$

Thus the real part is $e^{-\pi/2}$ and the imaginary part is 0.

b. Evaluate the real and imaginary parts of $(1 - j\sqrt{3})^{12}$.

Real part = 4096 Imaginary part = 0

The magnitude of $z = 1 - j\sqrt{3}$ is 2, and its phase angle is $-60°$. So the magnitude of z^{12} is $2^{12} = 4096$ and its phase is $-12 \times 60°$, which is a multiple of $360°$, and therefore the same as $0°$.

So $z^{12} = 4096$. Its real part is 4096 and its imaginary part is 0.

c. Express the real part of $e^{5j\theta}$ in terms of $\sin\theta$ and $\cos\theta$.

Real part = $\cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta$

From Euler's formula,

$$e^{5j\theta} = \cos 5\theta + j\sin 5\theta.$$

Its real part is $\cos 5\theta$. But we need to write it in terms of $\sin\theta$ and $\cos\theta$. So we better start with

$$e^{j\theta} = \cos\theta + j\sin\theta.$$

So

$$e^{5j\theta} = \left(e^{j\theta}\right)^5 = (\cos\theta + j\sin\theta.)^5.$$

The real part of the expansion is the odd-numbered terms, since those have only even powers of j. In order, they are

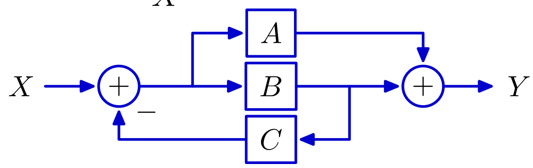
$$\cos^5\theta,$$
$$-10\cos^3\theta\sin^2\theta,$$
$$5\cos\theta\sin^4\theta.$$

So the real part, which is also $\cos 5\theta$, is

$$\cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta.$$

2. Yin-Yang

Determine the system functional $\frac{Y}{X}$ for the following system



where A, B, and C represent the system functionals for the boxed subsystems.

$$\frac{Y}{X} = \frac{A+B}{1+BC}$$

Let W represent the input to subsystem B. Then Black's equation can be used to show that

$$\frac{W}{X} = \frac{1}{1+BC}\,.$$

The output Y is the sum of AW and BW. So the result is

$$\frac{Y}{X} = \frac{A+B}{1+BC}\,.$$

3. Z transforms

Determine the Z transform (including the region of convergence) for each of the following signals:

a. $x_1[n] = \left(\frac{1}{2}\right)^n u[n-3]$

$X_1 =$	$\frac{1}{8z^2(z-\frac{1}{2})}$
ROC:	$\lvert z\rvert > \frac{1}{2}$

$$X_1(z) = \sum_n x_1[n]z^{-n} = \sum_{n=3}^{\infty}(1/2)^n z^{-n} = \sum_{n=3}^{\infty}\left(\frac{z^{-1}}{2}\right)^n .$$

Let $l = n-3$. Then

$$X_1(z) = \sum_{l=0}^{\infty}\left(\frac{z^{-1}}{2}\right)^{l+3} = \frac{(z^{-1}/2)^3}{1-(z^{-1}/2)} = \frac{1}{8z^2(z-\frac{1}{2})} .$$

The ROC is $|z| > 1/2$.

An alternative approach is to think of $x_1[n]$ as $\frac{1}{8}$ times a version of $\frac{1}{2}^n u[n]$ that is delayed by 3. The Z transform of $\frac{1}{2}^n u[n]$ is $\frac{z}{z-\frac{1}{2}}$. Delaying it by 3 multiplies the transform by z^{-3}; and scaling by $\frac{1}{8}$ scales the transform similarly.

b. $x_2[n] = (1+n)\left(\frac{1}{3}\right)^n u[n]$

$X_2 =$	$\frac{z^2}{\left(z-\frac{1}{3}\right)^2}$
ROC:	$\lvert z\rvert > \frac{1}{3}$

Start with the definition of the Z transform:

$$X(z) = \sum_{-\infty}^{\infty} x[n]z^{-n}$$

Differentiate both sides with respect to z:

$$\frac{dX(z)}{dz} = \sum_{-\infty}^{\infty} x[n](-n)z^{-n-1} = -z^{-1}\sum_{-\infty}^{\infty} nx[n]z^{-n}$$

This shows that

$$nx[n] \leftrightarrow -z\frac{dX(z)}{dz}$$

Since

$$\left(\frac{1}{3}\right)^n u[n] \leftrightarrow \frac{z}{z-\frac{1}{3}}, \quad |z| > \frac{1}{3}.$$

it follows that

$$n\left(\frac{1}{3}\right)^n u[n] \leftrightarrow -z\frac{d}{dz}\left(\frac{z}{z-\frac{1}{3}}\right) = \frac{\frac{1}{3}z}{(z-\frac{1}{3})^2}, \quad |z| > \frac{1}{3}.$$

$$X_2(z) = \frac{z}{z-\frac{1}{3}} + \frac{\frac{1}{3}z}{(z-\frac{1}{3})^2} = \frac{z^2}{(z-\frac{1}{3})^2}, \quad |z| > \frac{1}{3}$$

c. $x_3[n] = n\left(\frac{1}{2}\right)^{|n|}$

$X_3 =$ $\dfrac{3z(z-1)(z+1)}{2(z-2)^2(z-\frac{1}{2})^2}$

ROC: $\frac{1}{2} < |z| < 2$

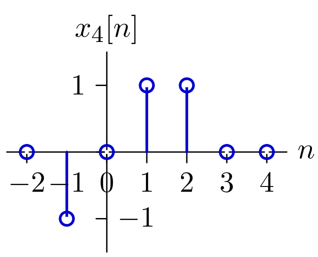
$$x_3[n] = n\left(\frac{1}{2}\right)^n u[n] + n\left(\frac{1}{2}\right)^{-n} u[-n-1]$$

$$n\left(\frac{1}{2}\right)^n u[n] \leftrightarrow \frac{\frac{1}{2}z}{(z-\frac{1}{2})^2}\,; \quad |z| > \frac{1}{2}.$$

$$n\left(\frac{1}{2}\right)^{-n} u[-n-1] \leftrightarrow \frac{-2z}{(z-2)^2}\,; \quad |z| < 2.$$

$$X_3(z) = \frac{\frac{1}{2}z}{(z-\frac{1}{2})^2} + \frac{-2z}{(z-2)^2} = -\frac{3z(z-1)(z+1)}{2(z-2)^2(z-\frac{1}{2})^2}\,; \quad \frac{1}{2} < |z| < 2.$$

d.



$X_4 =$ $-z + z^{-1} + z^{-2}$

ROC: $0 < |z| < \infty$

$$X_4(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} = x_4[-1]z + x_4[1]z^{-1} + x_4[2]z^{-2}$$
$$= -z + z^{-1} + z^{-2}; \quad 0 < |z| < \infty.$$

4. Inverse Z transforms

Determine all possible signals with Z transforms of the following forms.

a. $X_1(z) = \dfrac{1}{z-1}$

Enter expressions (or numbers) in the following table to discribe the possible signals. Each row should correspond to a different signal. If there are fewer signals than rows, enter **none** in the remaining rows.

	$n < -1$	$n = -1$	$n = 0$	$n = 1$	$n > 1$
$x_1[n] =$	-1	-1	-1	0	0
or	0	0	0	1	1
or	none	none	none	none	none

This transform has a pole at $z = 1$, as does the unit-step signal:

$$u[n] \leftrightarrow \sum_{n=0}^{\infty} z^{-n} = \frac{1}{1-z^{-1}} = \frac{z}{z-1}$$

Multiplying this transform by z^{-1} represents a unit delay in time:

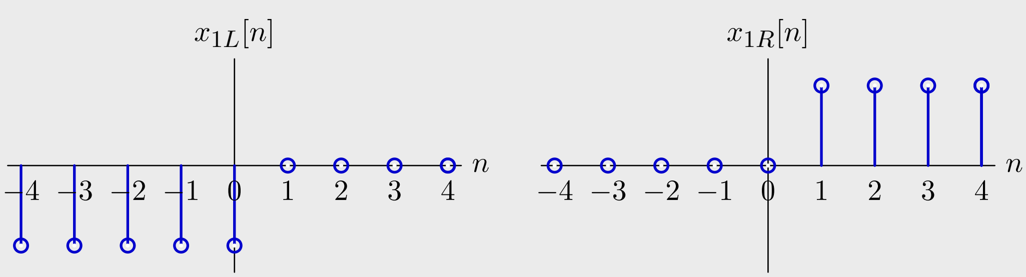
$$u[n-1] \leftrightarrow z^{-1}\sum_{n=0}^{\infty} z^{-n} = \frac{z^{-1}}{1-z^{-1}} = \frac{1}{z-1}$$

Thus there is a right-sided inverse transform

$$x_{1R}[n] = u[n-1]\,; \quad |z| > 1$$

and a left-sided inverse transform

$$x_{1L}[n] = -u[-n]\,; \quad |z| < 1\,.$$

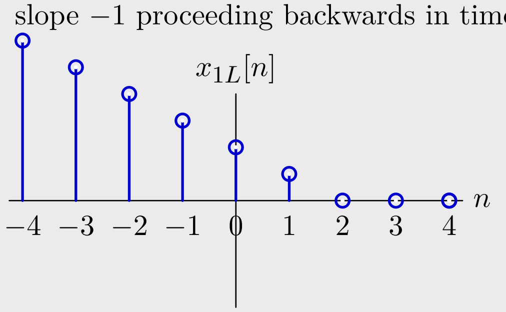


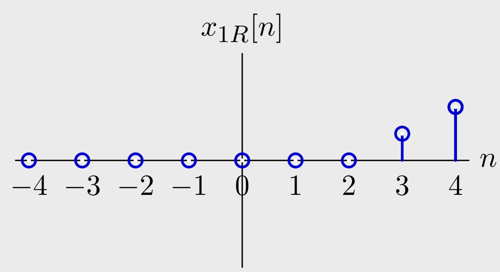
b. $X_2(z) = \dfrac{1}{z(z-1)^2}$

	$n < -1$	$n = -1$	$n = 0$	$n = 1$	$n > 1$
$x_2[n] =$	$2-n$	3	2	1	0
or	0	0	0	0	$n-2$
or	none	none	none	none	none

$$X_2(z) = z^{-2} \times \frac{z}{(z-1)^2}$$

The first term (z^{-2}) represents two units of delay. The second term corresponds to a unit ramp starting at $n = 0$. Because the ramp has a pole at 1, there are two regions of convergence. The right-sided region corresponds to a unit ramp starting at $n = 2$ (because of the two units of delay). The left-sided region corresponds to a ramp with slope -1 proceeding backwards in time from $n = 2$.





c. $X_3(z) = \dfrac{1}{z^2+z+1}$

	$n=-2$	$n=-1$	$n=0$	$n=1$	$n=2$
$x_3[n] =$	0	-1	1	0	0
or	0	0	0	0	1
or	none	none	none	none	none

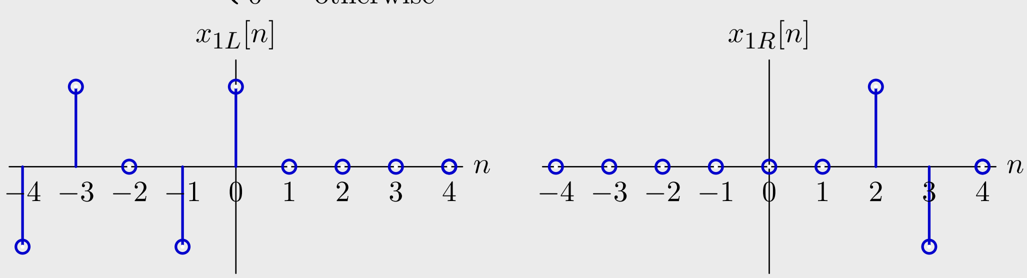
$$X_3(z) = \frac{1}{j\sqrt{3}}\left(\frac{1}{z-e^{j2\pi/3}} - \frac{1}{z-e^{-j2\pi/3}}\right) = \frac{1}{j\sqrt{3}}z^{-1}\left(\frac{z}{z-e^{j2\pi/3}} - \frac{z}{z-e^{-j2\pi/3}}\right)$$

Both terms correspond to geometric sequences. The base of the first is $e^{j2\pi/3}$, the base of the second is $e^{-j2\pi/3}$. The z^{-1} factor represents a unit of delay. Since the poles are on the unit circle, there are two regions of convergence: $|z|$ greater than 1 and less than 1. For $|z|>1$,

$$x_{3R}[n] = \frac{1}{j\sqrt{3}}\left(e^{j2(n-1)\pi/3} - e^{-j2(n-1)\pi/3}\right)u[n-1] = \frac{2}{\sqrt{3}}\sin\frac{2\pi(n-1)}{3}u[n-1]$$
$$= \begin{cases} 1 & n = 2,5,8,\ldots \\ -1 & n = 3,6,9\ldots \\ 0 & \text{otherwise} \end{cases}$$

For $|z|<1$,

$$x_{3L}[n] = -\frac{1}{j\sqrt{3}}\left(e^{j2(n-1)\pi/3} - e^{-j2(n-1)\pi/3}\right)u[-n] = -\frac{2}{\sqrt{3}}\sin\frac{2\pi(n-1)}{3}u[-n]$$
$$= \begin{cases} 1 & n = 0,-3,-6,-9,\ldots \\ -1 & n = -1,-4,-7\ldots \\ 0 & \text{otherwise} \end{cases}$$



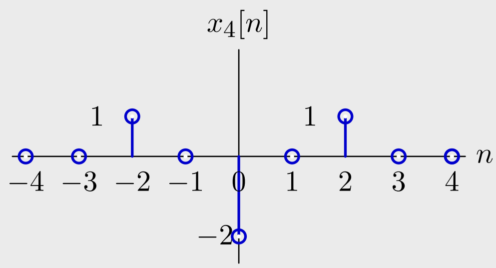
d. $X_4(z) = \left(\frac{1-z^2}{z}\right)^2$

	$n=-2$	$n=-1$	$n=0$	$n=1$	$n=2$
$x_4[n] =$	1	0	−2	0	1
or	none	none	none	none	none
or	none	none	none	none	none

$$X_4(z) = \frac{1-2z^2+z^4}{z^2} = z^{-2} - 2 + z^2$$

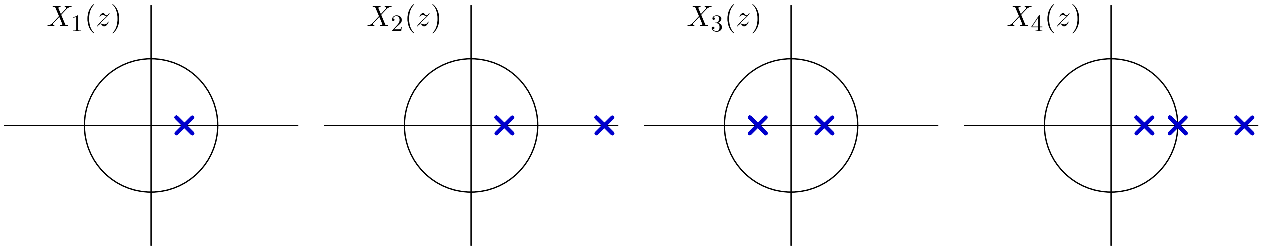
This sum converges for all z. Therefore there is a single region of convergence $0 < z < \infty$.

$$x_4[n] = \delta[n-2] - 2\delta[n] + \delta[n+2]$$

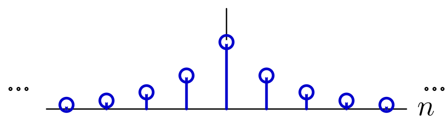


5. Poles

The following diagrams represent systems with poles (indicated by x's) but no zeros. The scale for each diagram is indicated by the circle, which has unit radius.



a. Which (if any) of $X_1(z)$ through $X_4(z)$ could represent a system with the following unit-sample response?



Enter a subset of the numbers 1 through 4 (separated by spaces) to represent $X_1(z)$ through $X_4(z)$ in the answer box below. If none of $X_1(z)$ through $X_4(z)$ apply, enter **none**.

1 and/or 2 and/or 3 and/or 4 or **none**: **none**

We can write the desired signal as

$$x[n] = a^{|n|}, \quad 0 < a < 1.$$

Then the Z-transform is given by:

$$X(z) = \sum_{n=0}^{\infty} a^n z^{-n} + \sum_{n=-\infty}^{-1} a^{-n} z^{-n}$$

$$= \frac{(a - \frac{1}{a})z}{(z-a)(z-\frac{1}{a})}\,; \quad a < |z| < \frac{1}{a}.$$

This Z-transform has poles at $z = a$, $z = \frac{1}{a}$, and a zero at $z = 0$. Therefore the answer is none of the above.

b. Which (if any) of the systems could be stable?

Hint: A system is stable iff the region of convergence of its Z transform includes the unit circle.[1]

1 and/or 2 and/or 3 and/or 4 or **none**: 1 2 3

System 1 will be stable if the region of convergence is outside the circle defined by the pole; system 2 will be stable if the region of convergence is the annulus defined by the two poles, and system 3 will be stable if the region of convergence is outside the circles defined by the poles. System 4 cannot be stable, since one of the poles is on the unit circle.

c. Which (if any) of systems could be causal?

Hint: A linear, time-invariant system is causal if its unit-sample response is zero for $t < 0$.

1 and/or 2 and/or 3 and/or 4 or **none**: 1 2 3 4

Any of the pole diagrams could represent such a system if the region of convergence is outside all of the circles defined by all of the poles.

d. Which (if any) of the systems could be both causal and stable?

1 and/or 2 and/or 3 and/or 4 or **none**: 1 3

For stability, the ROC must include the unit circle. A causal system must be right-sided. A right-sided system has and ROC of the form $|z| > a$. Only $X_1(z)$ and $X_3(z)$ represent such systems.

[1] If we decompose a system function using partial fractions, then we can consider the unit-sample response of the system as a sum of components that each correspond to one of the poles of the system. If the contribution of a pole is right-sided, then its Z transform converges for all z with magnitudes bigger than that of the pole. To be stable, the magnitude of that pole must be less than 1. It follows that the region of convergence includes the unit circle. A similar argument holds for left-sided contributions.

6. Periodic system

Consider this variant of the Fibonacci system:

$$y[n] = y[n-1] - y[n-2] + x[n]$$

where $x[n]$ represents the input and $y[n]$ represents the output.

a. Compute the unit-sample response and show that it is periodic. Enter the period in the box below.

period = 6

The system functional is

$$H = \frac{Y}{X} = \frac{1}{1 - \mathcal{R} + \mathcal{R}^2}\,.$$

One can expand the system functional using synthetic division as follows:

$$\begin{array}{r|llllllll}
 & 1 & +\mathcal{R} & +0\mathcal{R}^2 & -\mathcal{R}^3 & -\mathcal{R}^4 & +0\mathcal{R}^5 & +\mathcal{R}^6 & +\cdots \\ \hline
1 - \mathcal{R} + \mathcal{R}^2 & 1 & & & & & & & \\
 & 1 & -\mathcal{R} & +\mathcal{R}^2 & & & & & \\ \cline{2-4}
 & & \mathcal{R} & -\mathcal{R}^2 & & & & & \\
 & & \mathcal{R} & -\mathcal{R}^2 & +\mathcal{R}^3 & & & & \\ \cline{3-5}
 & & & & -\mathcal{R}^3 & & & & \\
 & & & & -\mathcal{R}^3 & +\mathcal{R}^4 & -\mathcal{R}^5 & & \\ \cline{5-7}
 & & & & & -\mathcal{R}^4 & +\mathcal{R}^5 & & \\
 & & & & & -\mathcal{R}^4 & +\mathcal{R}^5 & -\mathcal{R}^6 & \\ \cline{6-8}
 & & & & & & & +\mathcal{R}^6 & -+\cdots
\end{array}$$

The unit sample response is the sequence

$$h : 1, 1, 0, -1, -1, 0, 1, 1, 0, -1, -1, 0, \ldots$$

The form of this solution suggests that

$$\begin{aligned}
H &= 1 + \mathcal{R} - \mathcal{R}^3 - \mathcal{R}^4 + \mathcal{R}^6 + \mathcal{R}^7 - \mathcal{R}^9 - \mathcal{R}^{10} + + - - \cdots \\
&= (1 + \mathcal{R})(1 - \mathcal{R}^3)(1 + \mathcal{R}^6 + \mathcal{R}^{12} + \mathcal{R}^{18} + \cdots) \\
&= \frac{(1 + \mathcal{R})(1 - \mathcal{R}^3)}{1 - \mathcal{R}^6}
\end{aligned}$$

which is true because this expression equals the original system functional:

$$\frac{(1 + \mathcal{R})(1 - \mathcal{R}^3)}{1 - \mathcal{R}^6} = \frac{1}{1 - \mathcal{R} + \mathcal{R}^2}\,.$$

Equality can be seen by cross multiplication and expansion

$$(1 + \mathcal{R})(1 - \mathcal{R}^3)(1 - \mathcal{R} + \mathcal{R}^2) = 1 - \mathcal{R}^6\,.$$

It is clear from

$$H = \frac{(1 + \mathcal{R})(1 - \mathcal{R}^3)}{1 - \mathcal{R}^6}$$

that the unit-sample response is periodic and that the period is 6.

b. Enter the poles of the system in the box below (separated by spaces).

poles = $e^{j\pi/3}$ $e^{-j\pi/3}$

Substitute $\mathcal{R} = \frac{1}{z}$ in the system functional to get

$$\frac{z^2}{z^2 - z + 1}$$

and find the roots of the denominator:

$$z = \frac{1}{2} \pm \frac{j\sqrt{3}}{2} = e^{\pm j\pi/3}.$$

c. Decompose the system functional into partial fractions, and use the result to determine a closed-form expression for $h[n]$, the unit-sample response. Enter your expression in the box below.

$h[n] = \dfrac{\sin \pi(n+1)/3}{\sin \pi/3}$

$$\frac{1}{1 - \mathcal{R} + \mathcal{R}^2} = \frac{1}{j2\sin\frac{\pi}{3}} \left(\frac{e^{j\pi/3}}{1 - e^{j\pi/3}\mathcal{R}} - \frac{e^{-j\pi/3}}{1 - e^{-j\pi/3}\mathcal{R}} \right)$$

The corresponding unit-sample response is

$$h[n] = \frac{1}{j2\sin\frac{\pi}{3}} \left(e^{j\pi(n+1)/3} - e^{-j\pi(n+1)/3} \right) = \frac{\sin \pi(n+1)/3}{\sin \pi/3}$$

Engineering Design Problems

7. Scaling time

A system containing only adders, gains, and delays was designed with system functional

$$H = \frac{Y}{X}$$

which is a ratio of two polynomials in $\mathcal{R}$. When this system was constructed, users were dissatisfied with its responses. Engineers then designed three new systems, each based on a different idea for how to modify H to improve the responses.

System H_1: every delay element in H is replaced by a cascade of two delay elements.

System H_2: every delay element in H is replaced by a gain of $\frac{1}{2}$ followed by a delay.

System H_3: every delay element in H is replaced by a cascade of three delay elements.

For each of the following parts, evaluate the truth of the associated statement and enter **yes** if the statement is always true or **no** otherwise.

a. If H has a pole at $z = j = \sqrt{-1}$, then H_1 has a pole at $z = e^{j5\pi/4}$.

Yes or **No**: Yes

Explain.

The poles of H are the roots of the denominator of $H|_{\mathcal{R}\to\frac{1}{z}}$. But $H_1 = H|_{\mathcal{R}\to\mathcal{R}^2}$. Thus the poles of H_1 are the roots of the denominator of $H_1|_{\mathcal{R}\to\frac{1}{z}} = (H|_{R\to\mathcal{R}^2})|_{R\to\frac{1}{z}} = H|_{\mathcal{R}\to\frac{1}{z^2}}$. It follows that the poles of H_1 are the square roots of the poles of H.

If H has a pole at $z = j$ then H_1 must have poles at $z = \pm\sqrt{j}$. The two square roots of j are $e^{j\pi/4}$ and $e^{j5\pi/4}$. Thus $e^{j5\pi/4}$ is a pole of H_1.

b. If H has a pole at $z = p$ then H_2 has a pole at $z = 2p$.

Yes or **No**: No

Explain.

The poles of H are the roots of the denominator of $H|_{\mathcal{R}\to\frac{1}{z}}$. But $H_2 = H|_{\mathcal{R}\to\mathcal{R}/2}$. Thus the poles of H_2 are the roots of the denominator of $H_2|_{\mathcal{R}\to\frac{1}{z}} = \left(H|_{R\to\mathcal{R}/2}\right)\Big|_{R\to\frac{1}{z}} = H|_{\mathcal{R}\to\frac{1}{2z}}$. It follows that the poles of H_2 are half those of H.

If H has a pole at $z = p$ then H_2 must have poles at $z = p/2$ (not $2p$).

c. If H is stable then H_3 is also stable (where a system is said to be stable if all of its poles are inside the unit circle).

Yes or **No**: Yes

Explain.

The poles of H are the roots of the denominator of $H|_{\mathcal{R}\to\frac{1}{z}}$. But $H_3 = H|_{\mathcal{R}\to\mathcal{R}^3}$. Thus the poles of H_3 are the roots of the denominator of $H_3|_{\mathcal{R}\to\frac{1}{z}} = (H|_{R\to\mathcal{R}^3})|_{R\to\frac{1}{z}} = H|_{\mathcal{R}\to\frac{1}{z^3}}$. It follows that the poles of H_3 are the cube roots of the poles of H.

If H is stable, then the magnitudes of all of its poles are less than 1. It follows that the magnitudes of all of the poles of H_3 are also less than 1 since the magnitude of the cube root of a number that is less than 1 is also less than 1. Thus H_3 must also be stable.

8. Complex Sum

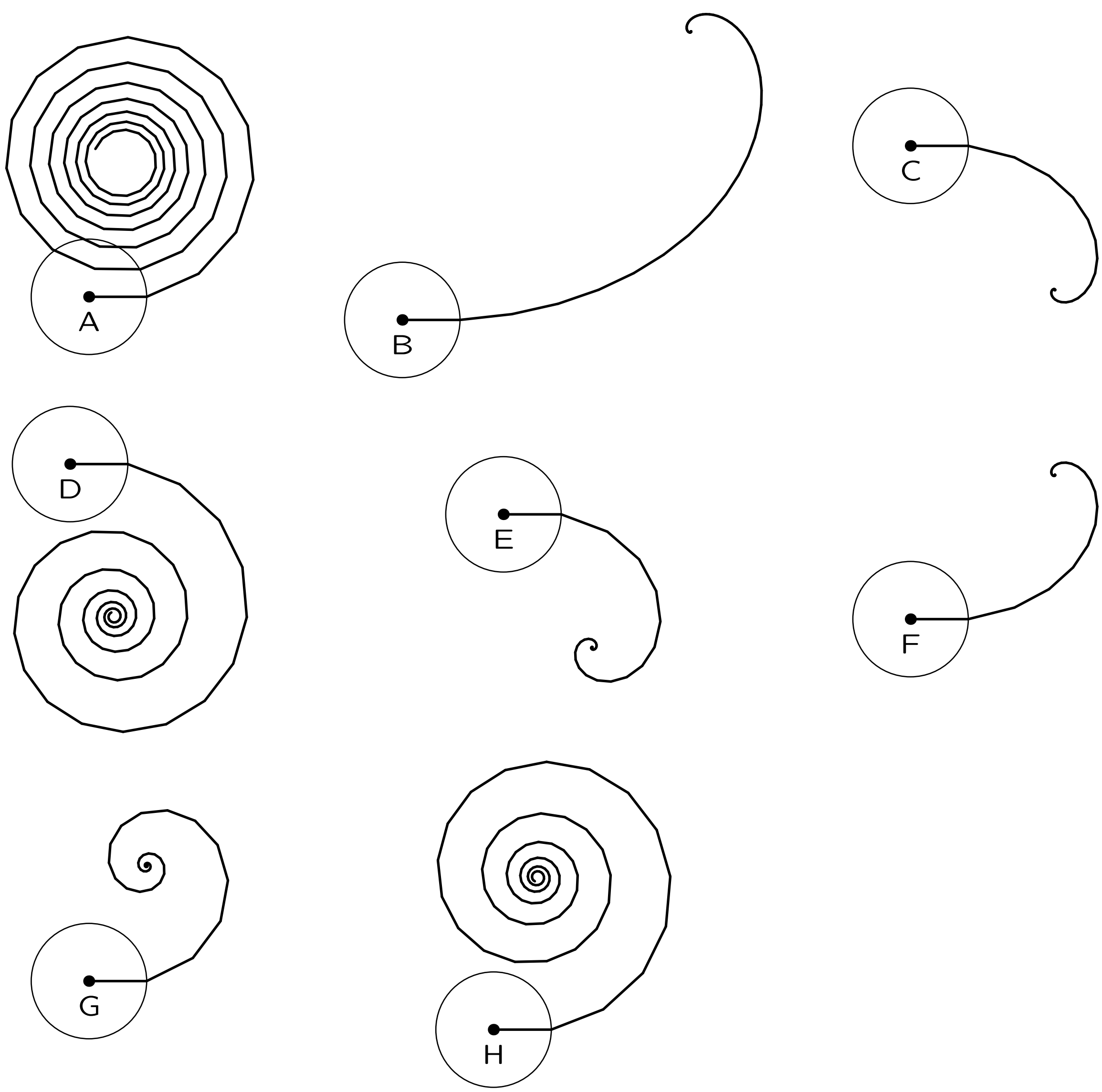
Each diagram below shows the unit circle in the complex plane, with the origin labeled with a dot.

Each diagram illustrates the sum

$$S = \sum_{n=0}^{100} \alpha^n.$$

Determine the diagram for which $\alpha = 0.8 + 0.2j$.

diagram = F



The first two terms in the series representation of S are 1 and $0.8+0.2j$. All of the curves start with a horizontal line to the right that stops at the intersection with the unit circle. Thus all of the curves correctly represent the first term. The second term should have a length that is somewhat shorter than 1 and an angle of $\tan^{-1}(0.25)$. We can immediately discard curves C, D, and E because their second terms have negative imaginary parts, which is wrong.

How quickly should the terms in the series converge? The magnitude of α is $\sqrt{0.8^2 + 0.2^2} \approx 0.8$. The magnitude of the last term in the sum is then approximately $0.8^{100} \approx \frac{1}{2}^{33} \approx \frac{1}{1000}^{3} < 10^{-9}$ — which would not be visible in the plots. Thus, the curve should appear to converge to a limit, which eliminates curves A and H. Furthermore, it says that the sum of 100 terms is very nearly equal to the infinite sum:

$$
\begin{aligned}
S \approx S_\infty = \sum_{n=0}^{\infty} \alpha^n &= \frac{1}{1-\alpha} = \frac{1}{1-(0.8+0.2j)} = \frac{1}{0.2-0.2j} \\
&= \frac{1}{0.2\sqrt{2}e^{-j\pi/4}} = \frac{5}{\sqrt{2}} e^{j\pi/4}
\end{aligned}
$$

Thus the final values of B and G are wrong, and the answer is F.

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