

Problem Set #3 Suggested Solutions

QUESTION 1:

Because technology is linear, the wage and the interest rate will be constant.

T = Lump-sum tax

t = payroll tax

w = wage (like an endowment in 2 periods since labor supply is inelastic)

b = benefits per elderly

g = debt per cohort

f = funding per cohort (some defined as per worker, which is fine, and will change the following equations, but the intuition should be the same)

Social Security Budget Constraint:

b = income from payroll tax + interest on assets in trust fund

$b = 2wt + rf$

The government has outstanding debt per capita, g, and we want to keep that constant over time. To do that, we collect lump-sum taxes from workers.

$$2T = rg \quad \text{or} \quad T = \frac{rg}{2}$$

Lifetime income:

$$I = [w(1-t) - T] + \frac{w(1-t) - T}{1+r} + \frac{b}{(1+r)^2} = [w(1-t) - \frac{rg}{2}] + \frac{w(1-t) - \frac{rg}{2}}{1+r} + \frac{2wt + rf}{(1+r)^2}$$

The (1- α) Lifecycle consumers:

Max $U[x,y,z] = xyz$

$$\text{Such that: } x + \frac{y}{1+r} + \frac{z}{(1+r)^2} = I$$

Setting marginal utilities equal, will want to set consumption in all periods equal, with discounting so:

$$x^* = \frac{y^*}{1+r} = \frac{z^*}{(1+r)^2} = \frac{I}{3}$$

For the α non-savers, we know that

$$x^* = y^* = w(1-t) - T$$

$$z^* = b = 2wt + rf$$

Market Clearance condition:

The key here is to determine how much private savings is in the economy. The current young are saving now, the current middle-aged are saving now, and have also earned interest on the 1st period's savings.

$$(k + g) = (1 - \alpha) \left\{ (2 + r) \left[w(1 - t) - \frac{rg}{2} - x^* \right] + \left[w(1 - t) - \frac{rg}{2} - y^* \right] \right\} + f$$

QUESTION 2:

Now the government is announcing a 1-time increase in the payroll tax rate. This was unanticipated, but it is known that it will only last 1 period. The revenue raised will permanently increase the amt of money in the trust fund, and then this will lead to a permanent lowering of the payroll tax later through the increased interest earned on the trust fund.

The first thing to note: the current elderly are “stuck” consuming what they have already saved and their SS benefit. There is no change in their budget constraint/lifetime earnings, so no changes in their consumption decisions from question 1. I leave them out of the analysis below because they aren’t saving and there are no changes to their behavior.

Government’s budget constraints:

In the initial period, the payroll tax is increased to $t_1 = t_0 + \Delta t$.

This revenue increases the trust fund, so starting next period, $f_2 = f_1 + 2w\Delta t$.

Use the increased interest to lower future payroll taxes, and keeping benefits constant:

$$b = 2wt + rf_1 = 2wt' + r(f_1 + 2w\Delta t)$$

$$t_2 = t_0 - r\Delta t$$

INITIAL PERIOD

Lifetime Income

These new budget constraints change the lifetime income for the young and the middle-aged differently.

Young:

$$I_y' = [w(1 - t_1) - T] + \frac{w(1 - t_2) - T}{1 + r} + \frac{b}{(1 + r)^2}$$

$$I_y' = [w(1 - t_0 - \Delta t) - \frac{rg}{2}] + \frac{w(1 - t_0 + r\Delta t) - \frac{rg}{2}}{1 + r} + \frac{2wt + rf}{(1 + r)^2}$$

Middle-aged:

$$I_M' = [w(1 - t_0) - T] + \frac{w(1 - t_1) - T}{1 + r} + \frac{b}{(1 + r)^2}$$

$$I_M' = [w(1 - t_0) - \frac{rg}{2}] + \frac{w(1 - t_0 - \Delta t) - \frac{rg}{2}}{1 + r} + \frac{2wt + rf}{(1 + r)^2}$$

(1-α) Life-cycle Consumers:

Young: Have 3 periods to maximize over:

$$\text{Max } U[x,y,z] = xyz$$

$$\text{Such that: } x + \frac{y}{1+r} + \frac{z}{(1+r)^2} = I_Y'$$

Setting marginal utilities equal, will want to set consumption in all periods equal, with discounting, so:

$$x_y^* = \frac{y_y^*}{1+r} = \frac{z_y^*}{(1+r)^2} = \frac{I_Y'}{3}$$

note: this is less than the optimal consumption chosen in question 1 because total income has gone down.

The middle-aged have only 2 periods to adjust their consumption to the new policy, since they have already made their consumption choice x^* for the first period.

$$\text{Max } U[x,y,z] = x^*yz$$

$$\text{ST. } x^* + \frac{y}{1+r} + \frac{z}{(1+r)^2} = I_M'$$

So now the optimal consumption is:

$$\frac{y_M^*}{1+r} = \frac{z^*}{(1+r)^2} = \frac{I_M' - x^*}{2}$$

Note: this is less than the y^* and z^* from question 1 due to the change in total income. This is also a bigger change in consumption than the young cohort realizes. Why? This cohort pays a higher payroll tax rate, but does not see any of the benefits from it because they stop working and don't get to pay the lower payroll tax rate later. So this cohort experiences a larger drop in lifetime income than the younger cohort does.

The α Non-savers:

The young:

$$x = w(1-t_1) - T = w(1-t_0 - \Delta t) - T$$

$$y = w(1-t_2) - T = w(1-t_0 + r\Delta t) - T$$

$$z = b$$

So the current young now have different consumption in every period. They consume less in their youth than in question 1, but more in their middle-age thanks to the lower payroll taxes later.

The middle aged:

$$x = w(1-t_0) - T \quad (\text{since you can't go back in time})$$

$$y = w(1-t_1) - T = w(1-t_0 - \Delta t) - T$$

$$z = b$$

So the current middle-aged enjoyed a relatively large consumption while they were young, and then experience the full drop in consumption today. They don't get any of the benefits of the future payroll tax deduction because they retire next year.

Market Clearance condition:

It is very important to keep track who is saving and when.

$$(k + g) =$$

$$(1 - \alpha) \{ [w(1-t_1) - T - x_y^*] + (1+r)[w(1-t_0) - T - x^*] + [w(1-t_1) - T - y_M^*] \} + f + 2w\Delta t$$

The first term is the savings of the current young. The second is the savings the current elderly did when they were young, and the third term is the saving they did in the current period. Finally, the public saving – including the increased saving from increasing the payroll tax.

Compared to Q1: The capital stock is larger. Why? The increase in the payroll tax decreases income. This increases the savings for those who save. Also, it is forcing savings (through the larger trust fund) from the non-savers.

IN EQUILIBRIUM:

Lifetime Savers:

Then all the savers set their consumption equal over all periods again. But the benefit and the tax formula have changed from that in question 1.

$$I'' = [w(1-t_2) - T] + \frac{w(1-t_2) - T}{1+r} + \frac{b}{(1+r)^2}$$

$$I'' = [w(1-t_0 + r\Delta t) - \frac{rg}{2}] + \frac{w(1-t_0 + r\Delta t) - \frac{rg}{2}}{1+r} + \frac{2wt + rf}{(1+r)^2}$$

$$x^* = \frac{y^*}{1+r} = \frac{z^*}{(1+r)^2} = \frac{I''}{3}$$

Since $I'' > I$, $I'' > I_M'$, and $I > I_y'$, consumption is higher now than in earlier scenarios.

Non-savers:

By definition, the non-savers consume all they can.

$$x^* = y^* = w(1-t_2) - T = w(1-t_0 + r\Delta t) - \frac{rg}{2}$$

$$z^* = b = 2wt + rf$$

x^* and y^* are higher than in either of the earlier scenarios, so these people are better off as well.

Market Clearance:

$$(k + g) = (1 - \alpha)\left\{(2 + r)\left[w(1 - t_2) - \frac{rg}{2} - x^*\right] + \left[w(1 - t_2) - \frac{rg}{2} - y^*\right]\right\} + f + 2w\Delta t$$

The capital stock is now LARGER than that in question 1, and LARGER than in the initial period.

Why?

Income is greater for everyone, leading to higher consumption. Savings also increases since the MPC for the savers is < 1 , and the public savings goes up through the larger trust fund (than in Q1).

QUESTION 3:

Now we add another government policy. In the initial period, the government also decreases the income tax, forever changing the debt per person.

$$t_1 = t_0 + \Delta t$$

$$t_2 = t_0 - r\Delta t$$

(same as above)

Now determine the income tax rates:

$$T_1 = \frac{rg_1}{2} - \theta w \Delta t$$

$$T_2 = \frac{rg_2}{2}$$

What is g_2 ?

$$g_1(1+r) - 2T_1 = g_2$$

$$g_2 = g_1 + 2\theta w \Delta t$$

$$\text{so } T_2 = \frac{r(g + 2\theta w \Delta t)}{2} = T_0 + \theta w r \Delta t$$

Initial Conditions:

Lifetime Income:

Young:

$$I_y''' = [w(1-t_1) - T_1] + \frac{w(1-t_2) - T_2}{1+r} + \frac{b}{(1+r)^2}$$

$$I_y''' = [w(1-t_0 - \Delta t) - \frac{rg}{2} + \theta w \Delta t] + \frac{w(1-t_0 + r\Delta t) - \frac{rg}{2} - \theta w r \Delta t}{1+r} + \frac{2wt + rf}{(1+r)^2}$$

Middle-aged:

$$I_M''' = [w(1-t_0) - T_0] + \frac{w(1-t_1) - T_1}{1+r} + \frac{b}{(1+r)^2}$$

$$I_M''' = [w(1-t_0) - \frac{rg}{2}] + \frac{w(1-t_0 - \Delta t) - \frac{rg}{2} + \theta w \Delta t}{1+r} + \frac{2wt + rf}{(1+r)^2}$$

Note: $I > I_y''' > I'' > I_y'$

And $I > I_M''' > I'' > I_M'$

(1-α) Life-cycle Consumers:

Young: Have 3 periods to maximize over:

$$\text{Max } U[x,y,z] = xyz$$

$$\text{Such that: } x + \frac{y}{1+r} + \frac{z}{(1+r)^2} = I_y'''$$

Setting marginal utilities equal, will want to set consumption in all periods equal, with discounting, so:

$$x^* = \frac{y^*}{1+r} = \frac{z^*}{(1+r)^2} = \frac{I_y'''}{3}$$

The middle-aged have only 2 periods to adjust their consumption to the new policy, since they have already made their consumption choice x^* for the first period.

$$\text{Max } U[x,y,z] = x^*yz$$

$$\text{ST. } x^* + \frac{y}{1+r} + \frac{z}{(1+r)^2} = I_M'''$$

So now the optimal consumption is:

$$\frac{y^*}{1+r} = \frac{z^*}{(1+r)^2} = \frac{I_M''' - x^*}{2}$$

Note: This is still a bigger change in consumption than the young cohort realizes.

The α Non-savers:

The young:

$$x = w(1-t_1) - T_1 = w(1-t_0 - \Delta t) - T_0 + \theta w \Delta t$$

$$y = w(1-t_2) - T_2 = w(1-t_0 + r \Delta t) - T_0 - \theta w \Delta t$$

$$z = b$$

So the current young still have different consumption in every period, but consumption in period 1 and 2 are closer than they were in question 2. They consume more in their youth than in question 2 because of the offsetting decrease in income tax, but less in their middle-age thanks to the higher income tax necessary to finance the increase in government debt.

The middle aged:

$$x = w(1-t_0) - T_0 \text{ (since you can't go back in time)}$$

$$y = w(1-t_1) - T_1 = w(1-t_0 - \Delta t) - T_0 + \theta w \Delta t$$

$$z = b$$

So the current middle-aged enjoyed a relatively large consumption while they were young, and then experience a drop in consumption today. They consume more now in middle-age than they did in question 2. They don't get any of the benefits of the future payroll tax deduction because they retire next year, nor do they get the additional cost of the higher debt payments.

Market Clearance condition:

It is very important to keep track who is saving and when.

$$(k + g + 2\theta w\Delta t) =$$

$$(1 - \alpha)\{[w(1 - t_1) - T_1 - x_y^*] + (1 + r)[w(1 - t_0) - T_0 - x^*] + [w(1 - t_1) - T_1 - y_M^*]\} + f + 2w\Delta t$$

Compared to initial period in Q2: The capital stock is $(1 - \theta)$ times the size, so the capital stock has decreased.

Compared to Q1: Still larger.

IN EQUILIBRIUM:

Lifetime Savers:

Then all the savers set their consumption equal over all periods again (with discounting).

But the benefit and the tax formula have changed from that in question 1.

$$I''' = [w(1 - t_2) - T_2] + \frac{w(1 - t_2) - T_2}{1 + r} + \frac{b}{(1 + r)^2}$$

$$I''' = [w(1 - t_0 + r\Delta t) - \frac{rg}{2} - \theta wr\Delta t] + \frac{w(1 - t_0 + r\Delta t) - \frac{rg}{2} - \theta wr\Delta t}{1 + r} + \frac{2wt + rf}{(1 + r)^2}$$

$$x^* = \frac{y^*}{1 + r} = \frac{z^*}{(1 + r)^2} = \frac{I'''}{3}$$

As long as $\theta < 1$, $I''' > I$, and consumption is higher now than in question 1.

As long as $\theta > 0$, $I''' < I''$, and consumption is lower now than in question 2 equilibrium.

Non-savers:

By definition, the non-savers consume all they can.

$$x^* = \frac{y^*}{1 + r} = w(1 - t_2) - T_2 = w(1 - t + r\Delta t) - \frac{rg}{2} - \theta wr\Delta t$$

$$z^* = b = 2wt + rf$$

Market Clearance:

$$(k + g + 2\theta w\Delta t) =$$

$$(1 - \alpha)\{(2 + r)[w(1 - t_2) - T_2 - x^*] + [w(1 - t_2) - T_2 - y^*]\} + f + 2w\Delta t$$

QUESTION 4

Now the non-savers pay the payroll tax and get SS benefits, and the savers only pay income tax.

So now the SS budget constraint is:

$\alpha b = 2\alpha t_0 w + r f$ since only paying to the non-savers, f has to be only αf_0 , and redefine f as funding per beneficiary in the cohort.

$$b = 2t_0 w + r f_0$$

And now the income tax condition changes because it is only being paid by the savers.

$$T_0 = \frac{r g_0}{2(1-\alpha)}$$

Now the policy changes:

$$t_1 = t_0 + \Delta t$$

$$t_2 = t_0 - r \Delta t$$

(same as above)

Now determine the income tax rates:

$$T_0 = \frac{r g_0}{2(1-\alpha)}$$

$$T_1 = T_0 - \Delta T$$

$$T_2 = T_0 + r \Delta T$$

as before....

What is ΔT ?

The amt of the change in taxes collected you want to offset = the amt will collect

$$\theta 2\alpha w \Delta t = 2(1-\alpha) \Delta T$$

$$\Delta T = \frac{\alpha}{1-\alpha} \theta w \Delta t$$

And based on the tax changes above, we know that:

$$f_1 = f_0 + 2\alpha w \Delta t \quad \text{and}$$

$$g_1 = g_0 + 2\alpha \theta w \Delta t$$

Initial period

Lifetime Income for savers:

Young:

$$I_y'''' = [w - T_1] + \frac{w(1-t_2) - T_2}{1+r}$$

$$I_y'''' = [w - \frac{rg}{2(1-\alpha)} + \frac{\alpha}{(1-\alpha)}\theta w\Delta t] + \frac{w - \frac{rg}{2(1-\alpha)} - \frac{\alpha}{(1-\alpha)}\theta w r\Delta t}{1+r}$$

Middle-aged:

$$I_M'''' = [w - T_0] + \frac{w - T_1}{1+r}$$

$$I_M'''' = [w - \frac{rg}{2(1-\alpha)}] + \frac{w - \frac{rg}{2(1-\alpha)} + \frac{\alpha}{(1-\alpha)}\theta w\Delta t}{1+r}$$

Note: the savers no longer pay payroll tax, and no longer get the benefit from it. This changes the timing of their income stream, AND their total income because they are no longer getting a return on the trust fund (on the non-savers forced saving).

With the policy change they experience a temporary decrease in income tax, followed by a permanent increase.

(1- α) Life-cycle Consumers:

Young: Have 3 periods to maximize over:

$$\text{Max } U[x,y,z] = xyz$$

$$\text{Such that: } x + \frac{y}{1+r} + \frac{z}{(1+r)^2} = I_y''''$$

Setting marginal utilities equal, will want to set consumption in all periods equal, with discounting, so:

$$x^* = \frac{y^*}{1+r} = \frac{z^*}{(1+r)^2} = \frac{I_y''''}{3}$$

The middle-aged have only 2 periods to adjust their consumption to the new policy, since they have already made their consumption choice x^* for the first period.

$$\text{Max } U[x,y,z] = x^*yz$$

$$\text{ST. } x^* + \frac{y}{1+r} + \frac{z}{(1+r)^2} = I_M''''$$

So now the optimal consumption is:

$$\frac{y^*}{1+r} = \frac{z^*}{(1+r)^2} = \frac{I_M'''' - x^*}{2}$$

The α Non-savers:

The young:

$$x = w(1-t_1) = w(1-t_0-\Delta t)$$

$$y = w(1-t_2) = w(1-t_0+r\Delta t)$$

$$z = b$$

The middle aged:

$$x = w(1-t_0) \quad (\text{since you can't go back in time})$$

$$y = w(1-t_1) = w(1-t_0-\Delta t)$$

$$z = b$$

The non-savers are better off than in Q3 because while they pay the same payroll tax, they are no longer paying any income tax, and the benefit in retirement is the same.

Market Clearance condition:

$$(k + g + 2\alpha\theta w\Delta t) =$$

$$(1-\alpha)\{[w - T_1 - x_y^*] + (1+r)[w - T_0 - x^*] + [w - T_1 - y_M^*]\} + f + 2\alpha w\Delta t$$

IN EQUILIBRIUM:

Lifetime Savers:

Then all the savers set their consumption equal over all periods again (with discounting).

But the benefit and the tax formula have changed from that in question 1.

$$I'''' = [w - T_2] + \frac{w - T_2}{1+r}$$

$$I'''' = [w - \frac{rg}{2(1-\alpha)} - \frac{\alpha}{(1-\alpha)}\theta wr\Delta t] + \frac{w - \frac{rg}{2(1-\alpha)} - \frac{\alpha}{1-\alpha}\theta wr\Delta t}{1+r}$$

$$x^* = \frac{y^*}{1+r} = \frac{z^*}{(1+r)^2} = \frac{I''''''}{3}$$

compared to the initial conditions, $I'''''' < I_M''''''$, so consumption falls

Non-savers:

By definition, the non-savers consume all they can.

$$x^* = \frac{y^*}{1+r} = w(1-t_2) = w(1-t+r\Delta t)$$

$$z^* = b = 2wt + rf$$

Compared to the initial conditions, consumption increases

Market Clearance:

$$(k + g + 2\theta\alpha w\Delta t) = (1-\alpha)\{(2+r)[w-T_2-x^*] + [w-T_2-y^*]\} + f + 2w\alpha\Delta t$$

Compared to the initial conditions, capital stock decreases because of a decrease in private savings due to the increased income taxes. But compared to pre-policy change, the effect is indeterminate. The public savings increases by $(1-\theta)\alpha w\Delta t$, while the private savings decreases by the $(1-\alpha) MPC*\Delta I$.

The main point of this question and problem set:

The incidence of taxes and the intergenerational issues depend heavily on who pays the tax, and what is happening to the capital stock (especially when you can have changes to wages and interest rates in the model). When everyone is covered by the SS system, any increase in taxes that *leads to smoother consumption* is good for the non-savers. Since the savers would smooth consumption through savings anyway, there is no effect on them except through changes in total income (since wage and interest rates don't change). But the economy grows because there is increased capital stock because public savings grows more than is offset by the decrease in private saving. And so future generations are made better off.

Now the change so only the non-savers pay the payroll tax, and only the savers pay the income tax. This is now a redistributive system w/in cohorts w/o any differences in wages. Increasing payroll taxes temporarily to increase the public saving leads to more income than before for the non-savers in the long-run. The effect on savers of offsetting this by increasing the public debt and increasing the income tax is to lower income and thus lower private savings. The overall effect depends on the number of people who are savers/non-savers and the amount of the payroll tax that is offset.