14.462 Advanced Maroeconomics
Spring 2003

Problem Set 2 Solution

Problem 1 (Human Capital and Incomplete Markets)

1. The Bellman equation is

$$V(w) = \max_{c,k,h,w'} \{U(c) + \beta \mathbb{E}[V(w')]\}$$
$$\text{s.t.} \;\; w' = \tilde{A}k^{\alpha} + \tilde{B}h^{\gamma} + R(w - c - k - h).$$

Notice that to simplify notation I do not give k and h a prime although they are next period variables. A natural guess for the value function is

$$V(w) = -\frac{1}{a\Gamma}\exp(-\Gamma(aw + b)).$$

In order to perform the optimization we need to evaluate $\mathbb{E}[V(w])$. We have

$$\mathbb{E}[w'] = \bar{A}k^{\alpha} + \bar{B}h^{\gamma} + R(w - c - k - h),$$
$$\text{Var}[w'] = \sigma_A^2 k^{2\alpha} + \sigma_B^2 h^{2\gamma} + 2\sigma_{AB}k^{\alpha}h^{\gamma}.$$

Now

$$\mathbb{E}[V(w')] = \mathbb{E}\left[-\frac{1}{a\Gamma}\exp(-\Gamma(aw' + b))\right]$$
$$= -\frac{1}{a\Gamma}\exp\left(-\Gamma(a\mathbb{E}[w'] + b) + \frac{1}{2}\Gamma^2 a^2 \text{Var}(w')\right).$$

Thus the objective in the Bellman equation can be written as

$$-\frac{1}{\Gamma}\exp(-\Gamma c) - \beta\frac{1}{a\Gamma}\exp\Bigg(- \Gamma a[\bar{A}k^{\alpha} + \bar{B}h^{\gamma} + R(w - c - k - h)] - \Gamma b$$
$$+ \frac{1}{2}\Gamma^2 a^2[\sigma_A^2 k^{2\alpha} + \sigma_B^2 h^{2\gamma} + 2\sigma_{AB}k^{\alpha}h^{\gamma}]\Bigg)$$

The first order condition with respect to consumption is given by

$$\exp(-\Gamma c) = \beta R \exp\Bigg(- \Gamma a[\bar{A}k^{\alpha} + \bar{B}h^{\gamma} + R(w - c - k - h)] - \Gamma b$$
$$+ \frac{1}{2}\Gamma^2 a^2[\sigma_A^2 k^{2\alpha} + \sigma_B^2 h^{2\gamma} + 2\sigma_{AB}k^{\alpha}h^{\gamma}]\Bigg)$$

The first order conditions with respect to k and h are

$$R = \alpha \bar{A} k^{\alpha-1} - \Gamma a \left(\alpha \sigma_A^2 k^{2\alpha-1} + \alpha \sigma_{AB} k^{\alpha-1} h^{\gamma} \right),$$
$$R = \gamma \bar{B} h^{\gamma-1} - \Gamma a \left(\gamma \sigma_B^2 h^{2\gamma-1} + \gamma \sigma_{AB} g^{\gamma-1} k^{\alpha} \right).$$

Notice that the level of wealth does not appear in these equations, so the optimal choice of k and h is independent of the state w. Denote the solutions to this system of equations as $k^*(a)$ and $h^*(a)$. Now define the function

$$\phi(a,b) = -[\Gamma(1+aR)]^{-1} \cdot \Big(\log(\beta R) - \Gamma a[\bar{A}k^*(a)^{\alpha} + \bar{B}h^*(a)^{\gamma} - R(k^*(a) + h^*(a))] - \Gamma b + \frac{1}{2}\Gamma^2 a^2 [\sigma_A^2 k^*(a)^{2\alpha} + \sigma_B^2 h^*(a)^{2\gamma} + 2\sigma_{AB} k^*(a)^{\alpha} h^*(a)^{\gamma}] \Big)$$

Then the first order condition for consumption can be rewritten as

$$\exp(-\Gamma c) = \exp(-\Gamma a R(w-c) - \Gamma(1+aR)\phi(a,b))$$

and so we obtain the consumption function

$$c = \frac{aR}{1+aR} w + \phi(a,b).$$

Substituting into the objective we get

$$\begin{aligned}
&-\frac{1}{\Gamma}\exp(-\Gamma c) - \frac{1}{\Gamma a R}\exp(-\Gamma a R(w-c) - \Gamma(1+aR)\phi(a,b)) \\
&= -\frac{1}{\Gamma}\exp\left(-\Gamma\left(\frac{aR}{1+aR}w + \phi(a,b)\right)\right) \\
&- \frac{1}{\Gamma a R}\exp\left(-\Gamma a R\left(w - \frac{aR}{1+aR}w - \phi(a,b)\right) - \Gamma(1+aR)\phi(a,b)\right) \\
&= -\frac{1}{\Gamma}\left(1 + \frac{1}{aR}\right)\exp\left(-\Gamma\left(\frac{aR}{1+aR}w + \phi(a,b)\right)\right) \\
&= -\frac{1}{\frac{aR}{1+aR}\Gamma}\exp\left(-\Gamma\left(\frac{aR}{1+aR}w + \phi(a,b)\right)\right)
\end{aligned}$$

For our guess to be correct this must be equal to $-\frac{1}{a\Gamma}\exp(-\Gamma(aw+b))$. Thus it must be the case that

$$a = \frac{aR}{1+aR}$$

which implies $a = \frac{R-1}{R}$. It must also be the case that $b = \phi(a,b)$, which yields

$$b = -[\Gamma a R]^{-1} \cdot \Big(\log(\beta R) - \Gamma a[\bar{A}k^*(a)^{\alpha} + \bar{B}h^*(a)^{\gamma} - R(k^*(a) + h^*(a))] + \frac{1}{2}\Gamma^2 a^2 [\sigma_A^2 k^*(a)^{2\alpha} + \sigma_B^2 h^*(a)^{2\gamma} + 2\sigma_{AB} k^*(a)^{\alpha} h^*(a)^{\gamma}] \Big)$$

Note that the consumption function can now be written as $c = aw + b$.
Now let's analyze how investment in physical and human captial depends on the variance and covariance of shocks. Totally differentiating the first order condition for physical capital yields

$$\begin{aligned} 0 = &[(\alpha-1)\alpha\bar{A}k^{\alpha-1} - \Gamma a\left((2\alpha-1)\alpha\sigma_A^2 k^{2\alpha-1} + (\alpha-1)\alpha\sigma_{AB}k^{\alpha-1}h^{\gamma}\right)]\frac{dk}{k} \\ &- \Gamma a\alpha\gamma\sigma_{AB}k^{\alpha-1}h^{\gamma-1}dh - \Gamma a\alpha k^{2\alpha-1}d\sigma_A^2 - \Gamma a\alpha k^{\alpha-1}h^{\gamma}d\sigma_{AB}. \end{aligned}$$

Using the first order condition to simplify the coefficient for dk we get

$$\begin{aligned} 0 = &-[(1-\alpha)Rk^{-1} + \Gamma a\alpha^2\sigma_A^2 k^{2(\alpha-1)}]dk - \Gamma a\alpha\gamma\sigma_{AB}k^{\alpha-1}h^{\gamma-1}dh \\ &- \Gamma a\alpha k^{2\alpha-1}d\sigma_A^2 - \Gamma a\alpha k^{\alpha-1}h^{\gamma}d\sigma_{AB}. \end{aligned}$$

Dividing by $\Gamma a\alpha k^{\alpha-1}$ and rearranging we get

$$[\psi_k + \alpha\sigma_A^2 k^{\alpha-1}]dk + \gamma\sigma_{AB}h^{\gamma-1}dh = -k^{\alpha}d\sigma_A^2 - h^{\gamma}d\sigma_{AB}.$$

where $\psi_k = \frac{1-\alpha}{\alpha}\frac{1}{\Gamma}\frac{1}{k^{\alpha}}\frac{R^2}{R-1} > 0$.
By symmetry

$$[\psi_h + \gamma\sigma_B^2 h^{\gamma-1}]dh + \alpha\sigma_{AB}k^{\alpha-1}dk = -h^{\gamma}d\sigma_B^2 - k^{\alpha}d\sigma_{AB}.$$

where $\psi_h = \frac{1-\gamma}{\gamma}\frac{1}{\Gamma}\frac{1}{k^{\alpha}}\frac{R^2}{R-1} > 0$.
Writing this system in matrix form we have

$$\begin{bmatrix} \psi_k + \alpha\sigma_A^2 k^{\alpha-1} & \gamma\sigma_{AB}h^{\gamma-1} \\ \alpha\sigma_{AB}k^{\alpha-1} & \psi_h + \gamma\sigma_B^2 h^{\gamma-1} \end{bmatrix} \cdot \begin{bmatrix} dk \\ dh \end{bmatrix} = \begin{bmatrix} -k^{\alpha}d\sigma_A^2 - h^{\gamma}d\sigma_{AB} \\ -h^{\gamma}d\sigma_B^2 - k^{\alpha}d\sigma_{AB} \end{bmatrix}$$

Then

$$\begin{aligned} \begin{bmatrix} dk \\ dh \end{bmatrix} = &\frac{1}{\psi_k\psi_h + \psi_k\gamma\sigma_B^2 h^{\gamma-1} + \psi_h\alpha\sigma_A^2 k^{\alpha-1} + \alpha k^{\alpha-1}\gamma h^{\gamma-1}[\sigma_A^2\sigma_B^2 - \sigma_{AB}^2]} \\ &\cdot \begin{bmatrix} \psi_h + \gamma\sigma_B^2 h^{\gamma-1} & -\gamma\sigma_{AB}h^{\gamma-1} \\ -\alpha\sigma_{AB}k^{\alpha-1} & \psi_k + \alpha\sigma_A^2 k^{\alpha-1} \end{bmatrix} \cdot \begin{bmatrix} -k^{\alpha}d\sigma_A^2 - h^{\gamma}d\sigma_{AB} \\ -h^{\gamma}d\sigma_B^2 - k^{\alpha}d\sigma_{AB} \end{bmatrix} \\ = &\frac{1}{\psi_k\psi_h + \psi_k\gamma\sigma_B^2 h^{\gamma-1} + \psi_h\alpha\sigma_A^2 k^{\alpha-1} + \alpha k^{\alpha-1}\gamma h^{\gamma-1}[\sigma_A^2\sigma_B^2 - \sigma_{AB}^2]} \\ &\cdot \begin{bmatrix} -(\psi_h + \gamma\sigma_B^2 h^{\gamma-1})k^{\alpha}d\sigma_A^2 + \gamma\sigma_{AB}h^{2\gamma-1}d\sigma_B^2 - (\psi_h h^{\gamma} + \gamma h^{\gamma-1}(\sigma_B^2 h^{\gamma} - \sigma_{AB}k^{\alpha}))d\sigma_{AB} \\ -(\psi_k + \alpha\sigma_A^2 k^{\alpha-1})h^{\gamma}d\sigma_B^2 + \alpha\sigma_{AB}k^{2\alpha-1}d\sigma_A^2 - (\psi_k k^{\alpha} + \alpha k^{\alpha-1}(\sigma_A^2 k^{\alpha} - \sigma_{AB}h^{\gamma}))d\sigma_{AB} \end{bmatrix} \end{aligned}$$

The term $\sigma_A^2\sigma_B^2 - \sigma_{AB}^2$ is nonnegative as it is the determinant of the variance-covariance matrix (Cauchy-Schwartz inequality). Thus the determinant is positive.
We find that $\frac{dk}{d\sigma_A^2}$ and $\frac{dh}{d\sigma_B^2}$ are both negative. An increase in the riskiness of the own

return unambigously reduces investment. Notice that $\frac{dk}{d\sigma_B^2}$ and $\frac{dh}{d\sigma_A^2}$ are positive if $\sigma_{AB} > 0$ and negative if $\sigma_{AB} < 0$. If human capital becomes more risky and the two returns are positively correlated, then physical capital is relatively more attractive and investment in physical capital increases. Now suppose that the correlation of the returns is negative and human capital becomes more risky. Investment in human capital shrinks. As human capital provides insurance for physical capital, the amount of insurance of physical capital shrinks and so physical capital investment is reduced as well.

2. With $\sigma_{AB} = 0$ and $\alpha = \gamma = \frac{1}{2}$, the first order condition for k reduces to

$$R = \frac{1}{2}\bar{A}k^{-\frac{1}{2}} - \Gamma\left(\frac{R-1}{R}\right)\frac{1}{2}\sigma_A^2$$

and solving for k we obtain

$$k = \left[\frac{R\bar{A}}{2R + (R-1)\Gamma\sigma_A^2}\right]^2$$

and by symmetry

$$h = \left[\frac{R\bar{B}}{2R + (R-1)\Gamma\sigma_B^2}\right]^2$$

3. When $\sigma_{AB} = 0$, investment in the two different capital stocks is independent.

4. From our previous calculations we know that if $\sigma_{AB} = 0$, $\sigma_A = 0$ and $\sigma_B > 0$, then an increase in σ_B leaves investment in physical capital unaffected and reduces investment in human capital.
To derive the Euler equation, first compute the envelope condition as $V'(w) = \beta R\mathbb{E}[V'(w')]$ and recall that the first order condition for consumption is $U'(c) = \beta R\mathbb{E}[V'(w')]$. Combining these two equations we have $V'(w) = U'(c)$ and thus

$$U'(c) = \beta R\mathbb{E}[U'(c')].$$

With our functional form for U this becomes

$$\exp(-\Gamma c) = \beta R\mathbb{E}[\exp(-\Gamma c')]$$

As $c' = aw' + b$ it is normally distributed and so

$$\exp(-\Gamma c) = \beta R\exp\left(-\Gamma\mathbb{E}[c'] + \frac{1}{2}\Gamma^2\mathrm{Var}(c')\right).$$

Taking logs and rearranging yields

$$\mathbb{E}[c'] - c = \frac{\log(\beta R)}{\Gamma} + \frac{\Gamma}{2}\mathrm{Var}(c').$$

Moreover $\mathrm{Var}(c') = \mathrm{Var}(aw' + b) = a^2\mathrm{Var}(w')$ and with $\sigma_{AB} = 0$ and $\sigma_A = 0$ we have $\mathrm{Var}(w') = \sigma_B^2 h^{2\gamma}$. Using these facts the Euler equation can be rewritten as

$$\mathbb{E}[c'] - c = \frac{\log(\beta R)}{\Gamma} + \frac{\Gamma}{2}\left(\frac{R-1}{R}\right)^2 \sigma_B^2 h^{2\gamma}.$$

Due to the linearity of the Euler equation in $\mathbb{E}[c']$ and c and the fact that all households invest the same amount $h = H$ we can aggregate across households to obtain

$$C' - C = \frac{\log(\beta R)}{\Gamma} + \frac{\Gamma}{2}\left(\frac{R-1}{R}\right)^2 \sigma_B^2 H^{2\gamma}.$$

The expectation disappears as aggregate consumption is deterministic. In steady state we must have

$$0 = \log(\beta R) + \frac{\Gamma^2}{2}\left(\frac{R-1}{R}\right)^2 \sigma_B^2 H^{2\gamma}$$

This together with the first order equation for human capital

$$0 = \gamma \bar{B} H^{\gamma-1} - \Gamma \frac{R-1}{R} \gamma \sigma_B^2 - R$$

determines R and H in steady state. We want to know how this depends on σ_B^2. Note that σ_B^2 enters these equations in two places. In the Euler equation idiosyncratic risk in human capital investment creates a precautionary savings effect. In the first order condition for human capital in exerts a direct negative investment effect.
The effect of σ_B^2 on R and H is ambiguous. Let's consider the scenario which Angeletos and Calvet would consider the most likely. So suppose from now on that in the Euler equation the direct effect precautionary savings effect of σ_B^2 on R wins out and R is decreasing in σ_B^2. From the first oder equation for human capital we can write $H = \tilde{H}(R, \sigma_B^2)$. The direct investment effect tends to reduce investment. But through the precautionary savings effect σ_B^2 reduces the interest rate which tends to increase investment in human capital. Investment in human capital will fall if the investment effect dominates. Investment in physical captial will increase with the reduction in the interest rate caused by the precautionary savings effect, so the general equilbrium effect of an increase in the idiosyncratic risk in human capital is to increase investment in physical capital.

5. The human capital externality affects the first order condition for capital investment, which now becomes

$$R = \alpha \bar{A} H^{\eta} k^{\alpha-1}$$

Maintaining our assumptions about which effects dominate from the previous part, an increase in σ_B^2 reduces both R and H. Thus we have two opposing effects on investment in physical capital. The precautionary savings effect tends to increase investment in physical capital, but the reduction in human capital investment tends to reduce physical capital investment through the externality.

6. If labor income risk corresponds to riskiness in human capital investments, and if human capital externalities are strong, then physical capital accumulation is likely to be reduced.

Problem 2 (based on Kiyotaki and Moore (1997), Midterm Exam 2000)

1. Before solving the problem I'll discuss where equation (3) comes from. If a gatherer buys a unit of land, he pays q_t today. His output tomorrow increases by $G'(k_t')$ and the unit of land can be resold for q_{t+1}. As the discount factor is R^{-1}, we get the condition

$$q_t = \frac{G'(k_t') + q_{t+1}}{R}$$

Now using the market clearing condition $k_t + k_t' = \bar{K}$ to replace k_t' and rearraning slightly yields the first equality of equation (3). The second equality is simply a convenient functional form assumption. Farmers are always eager to expand, so the borrowing constraint will always be binding unless there are surprises. In particular the borrowing constraint will be binding in steady state. The steady state versions of (1),(2) and (3) are

$$\begin{aligned} Rb &= qk \\ 0 &= ak + b - Rb \\ q - \frac{1}{R}q &= \alpha k^{\frac{1}{\eta}} \end{aligned}$$

The first equation gives $b = \frac{qk}{R}$. Substituting into the second equation yields

$$ak = \frac{R-1}{R}qk,$$

so the steady state price of land is $q^* = a\frac{R}{R-1}$. Substituting this result into the third equation yields

$$a = \alpha k^{\frac{1}{\eta}}$$

so steady state land holdings of farmers are $k^* = \left(\frac{a}{\alpha}\right)^{\eta}$. Plugging these results into the first equation yields

$$b^* = \frac{q^*k^*}{R} = \frac{1}{R}\frac{R}{R-1}a\left(\frac{a}{\alpha}\right)^{\eta} = \frac{1}{R-1}a^{1+\eta}\alpha^{-\eta}.$$

2. Let k_1^*, q_1^* and b_1^* be the steady state corresponding to a_1. Similarly let k_2^*, q_2^* and b_2^* be the steady state corresponding to a_2.
Consider some date $t \geq s$. As farmers are eager to expand and their are no more surprises the borrowing constraint will always be binding:

$$Rb_s = q_{s+1}k_s.$$

Substituing into the flow of funds constraint of farmers we get

$$q_s(k_s - k_{s-1}) = a_s k_{s-1} + \frac{q_{s+1}k_s}{R} - Rb_{s-1}$$

or slightly rearranged

$$\left(q_s - \frac{q_{s+1}}{R}\right) k_s = (a_s + q_s)k_{s-1} - Rb_{s-1}.$$

Substituting for $q_s - \frac{q_{s+1}}{R}$ from equation (3) we get

$$\alpha k_s^{\frac{1}{\eta}} k_s = (a_s + q_s)k_{s-1} - Rb_{s-1}$$

or equivalently

$$k_s = \left(\frac{(a_s + q_s)k_{s-1} - Rb_{s-1}}{\alpha}\right)^{\frac{\eta}{1+\eta}}.$$

At time t this equation becomes

$$k_t = \left(\frac{(a_2 + q_t)k_1^* - Rb_1^*}{\alpha}\right)^{\frac{\eta}{1+\eta}}.$$

and as $Rb_1^* = q_1^* k_1^*$ this can also be written as

$$k_t = \left(\frac{(a_2 + q_t - q_1^*)k_1^*}{\alpha}\right)^{\frac{\eta}{1+\eta}}.$$

For $s \geq t+1$ the borrowing constraint was also binding in the previous period, so

$$\begin{aligned} k_s &= \left(\frac{(a_2 + q_s)k_{s-1} - q_s k_{s-1}}{\alpha}\right)^{\frac{\eta}{1+\eta}} \\ &= \left(\frac{a_2 k_{s-1}}{\alpha}\right)^{\frac{\eta}{1+\eta}} \end{aligned}$$

Collecting equations we have

$$
\begin{aligned}
k_t &= \left(\frac{(a_2 + q_t - q_1^*)k_1^*}{\alpha}\right)^{\frac{\eta}{1+\eta}}, \\
k_s &= \left(\frac{a_2 k_{s-1}}{\alpha}\right)^{\frac{\eta}{1+\eta}} \qquad \forall \ s \geq t+1, \\
q_s &= \frac{1}{R} q_{s+1} + \alpha k_s^{\frac{1}{\eta}} \qquad \forall \ s \geq t.
\end{aligned}
$$

The next step is to log-linearize around the new steady state. For a variable x_s define $\check{x}_s = \log(x_s) - \log(x_2)$. Also define $\Delta = \log(a_2) - \log(a_1)$. Let's begin with the first equation. Using the identity $k_2^* = \left(\frac{a_2}{\alpha} k_2^*\right)^{\frac{\eta}{1+\eta}}$ we can write it as

$$
\frac{k_t}{k_2^*} = \left(\frac{a_2 + q_t - q_1^*}{a_2} \frac{k_1^*}{k_2^*}\right)^{\frac{\eta}{1+\eta}}
$$

or

$$
\left(\frac{k_t}{k_2^*}\right)^{1+\frac{1}{\eta}} = \left(1 + \frac{q_t - q_1^*}{a_2}\right) \frac{k_1^*}{k_2^*}
$$

Taking logs, noting that $\frac{k_1^*}{k_2^*} = \left(\frac{a_1}{a_2}\right)^{\eta}$ we get that approximately

$$
\left(1 + \frac{1}{\eta}\right) \check{k}_t = \frac{q_t - q_1^*}{a_2} - \eta\Delta
$$

Using the facts that $q_2^* = a_2 \frac{R}{R-1}$ and $\frac{q_2^*}{q_1^*} = \frac{a_2}{a_1}$ we have that approximately

$$
\frac{q_t - q_1^*}{a_2} = \frac{R}{R-1}\left(\frac{q_t - q_2^*}{q_2^*} + \frac{q_2^* - q_1^*}{q_2^*}\right) = \frac{R}{R-1}(\check{q}_t + \Delta).
$$

Thus the log-linearized version of the first equation is given by

$$
\left(1 + \frac{1}{\eta}\right) \check{k}_t = \frac{R}{R-1}\check{q}_t + \left(\frac{R}{R-1} - \eta\right)\Delta.
$$

For the second equation we simply get

$$
\check{k}_s = \frac{\eta}{1+\eta} \check{k}_{s-1} \qquad \forall \ s \geq t+1.
$$

Using the identity $q_2^* = \frac{1}{R} q_2^* + \alpha (k_2^*)^{\frac{1}{\eta}}$ the third equation can be rewritten as

$$
\frac{q_s - q_2^*}{q_2^*} = \frac{1}{R}\frac{q_{s+1} - q_2^*}{q_2^*} + \frac{1}{q_2^*}\alpha\left(k_s^{\frac{1}{\eta}} - (k_2^*)^{\frac{1}{\eta}}\right).
$$

and so approximately

$$\check{q}_s = \frac{1}{R}\check{q}_{s+1} + \frac{1}{q_2^*}\alpha\frac{1}{\eta}(k_2^*)^{\frac{1}{\eta}}\left(\frac{k_s - k_2^*}{k_2^*}\right)$$

Using the facts that $q_2^* = \frac{R}{R-1}a_2$ and $(k_2^*)^{\frac{1}{\eta}} = \frac{a_2}{\alpha}$ we see that the log-linearized version of the third equation is

$$\check{q}_s = \frac{1}{R}\check{q}_{s+1} + \frac{1}{\eta}\frac{R-1}{R}\check{k}_s \qquad \forall\ \ s \geq t.$$

Collecting our results, the log-linearized system is

$$\begin{aligned}
\left(1+\frac{1}{\eta}\right)\check{k}_t &= \frac{R}{R-1}\check{q}_t + \left(\frac{R}{R-1} - \eta\right)\Delta, \\
\check{k}_s &= \frac{\eta}{1+\eta}\check{k}_{s-1} \qquad \forall\ \ s \geq t+1, \\
\check{q}_s &= \frac{1}{R}\check{q}_{s+1} + \frac{1}{\eta}\frac{R-1}{R}\check{k}_s \qquad \forall\ \ s \geq t.
\end{aligned}$$

By induction we obtain from the second equation that

$$\check{k}_s = \left(\frac{\eta}{1+\eta}\right)^{s-t}\check{k}_t$$

and iterating the third equation imposing $\lim_{s\to\infty} R^{-(s-t)}\check{q}_s = 0$ yields

$$\check{q}_t = \sum_{s=t}^{\infty}\frac{1}{R^{s-t}}\left(\frac{1}{\eta}\frac{R-1}{R}\check{k}_s\right).$$

Combining the last two results gives

$$\check{q}_t = \frac{\frac{1}{\eta}\frac{R-1}{R}}{1-\frac{1}{R}\frac{\eta}{1+\eta}}\check{k}_t.$$

Together with the first equation of the log-linearized system we now have two equations in the two unknowns $\check{q}_t$ and $\hat{q}_t$. To solve, rewrite the last equation as

$$\left(1-\frac{1}{R}\frac{\eta}{1+\eta}\right)\check{q}_t = \frac{1}{1+\eta}\frac{R-1}{R}\left(1+\frac{1}{\eta}\right)\check{k}_t.$$

Substituting from the first equation of the log-linearized system this becomes

$$\left(1-\frac{1}{R}\frac{\eta}{1+\eta}\right)\check{q}_t = \frac{1}{1+\eta}\left[\check{q}_t + \left(1-\frac{R-1}{R}\eta\right)\Delta\right]$$

This can be rewritten as

$$\frac{R-1}{R}\frac{\eta}{1+\eta}\check{q}_t = \frac{1}{1+\eta}\left(1-\frac{R-1}{R}\eta\right)\Delta$$

and so we get

$$\check{q}_t = \left(\frac{R}{R-1}\frac{1}{\eta}-1\right)\Delta$$

and consequently

$$\begin{aligned}
\check{k}_t &= \frac{R}{R-1}\eta\left(1-\frac{1}{R}\frac{\eta}{1+\eta}\right)\left(\frac{R}{R-1}\frac{1}{\eta}-1\right)\Delta \\
&= \frac{\eta}{1+\eta}\left((1+\eta)\frac{R}{R-1}-\frac{1}{R-1}\eta\right)\left(\frac{R}{R-1}\frac{1}{\eta}-1\right)\Delta \\
&= \frac{\eta}{1+\eta}\left(\frac{R}{R-1}+\eta\right)\left(\frac{R}{R-1}\frac{1}{\eta}-1\right)\Delta
\end{aligned}$$

Notice that there will be overshooting if $\frac{R}{R-1} > \eta$, i.e. if the residual supply of land to farmers is not very elastic.

Problem 3 (Savings with Incomplete Markets, General Exam 2001)

1. If individual h specializes in storage, then

$$\begin{aligned}
\mathbb{E}_t U_t(h) &= \log(1-k_t(h)) + \beta\mathbb{E}_t[\log(\delta k_t(h))] \\
&= \log(1-k_t(h)) + \beta\log(k_t(h)) + \beta\log(\delta).
\end{aligned}$$

If instead the individual specializes in the risky technology, then

$$\begin{aligned}
\mathbb{E}_t U_t(h) &= \log(1-k_t(h)) + \beta\mathbb{E}_t[\log(\tilde{A}_{t+1}(h)k_t(h))] \\
&= \log(1-k_t(h)) + \beta\log(k_t(h)) + \beta\left[\frac{1}{2}\log(A+\sigma)+\frac{1}{2}\log(A-\sigma)\right].
\end{aligned}$$

2. In words, $\hat{\sigma}$ is the counterfactual riskiness of the risky project that makes the certainty equivalent of the return to the risky project equal to the return of the storage technology. If we had $\sigma = \hat{\sigma}$, then the individual would be indifferent between the risky project and the storage technology. So $\hat{\sigma}$ is implicitly defined by the equation

$$\frac{1}{2}\log(A+\hat{\sigma})+\frac{1}{2}\log(A-\hat{\sigma}) = \log(\delta).$$

As $A > \delta$ we know that $\hat{\sigma} > 0$ by Jensen's inequality. Totally differentiating yields

$$\frac{1}{2}\left[\frac{1}{A+\hat{\sigma}} - \frac{1}{A-\hat{\sigma}}\right] d\hat{\sigma} + \frac{1}{2}\left[\frac{1}{A+\hat{\sigma}} + \frac{1}{A-\hat{\sigma}}\right] dA = \frac{d\delta}{\delta}$$

which can be rewritten as

$$-\frac{\hat{\sigma}}{(A+\hat{\sigma})(A-\hat{\sigma})} d\hat{\sigma} + \frac{A}{(A+\hat{\sigma})(A-\hat{\sigma})} dA = \frac{d\delta}{\delta}.$$

Thus

$$\frac{\partial \hat{\sigma}}{\partial \delta} = -\frac{(A+\hat{\sigma})(A-\hat{\sigma})}{\delta \hat{\sigma}} < 0$$

and

$$\frac{\partial \hat{\sigma}}{\partial A} = \frac{A}{\hat{\sigma}} > 0.$$

If the storage technology has a better return, then the risky project must become less risky to keep the individual indifferent. If the average return of the risky project increases, then it must become more risky to keep the individual indifferent.
The definition of the certainty equivalent is

$$B = \exp(\mathbb{E}(\log \tilde{A})) = \exp\left(\frac{1}{2}\log(A+\sigma) + \frac{1}{2}\log(A-\sigma)\right) = \sqrt{(A+\sigma)(A-\sigma)}.$$

Clearly $\frac{\partial B}{\partial \sigma} < 0$ and $\frac{\partial B}{\partial A} > 0$. Making the project more risky reduces the certainty equivalent of its return. Increasing the average return of the project increases the certainty equivalent.
Now consider B as a function of σ, writing $B(\sigma)$. Then $B(0) = A$, $B(\hat{\sigma}) = \delta$ and $B(A) = 0$.
The individual specializes in the risky technology if $\sigma < \hat{\sigma}$ and in storage if $\sigma > \hat{\sigma}$, being indifferent if $\sigma = \hat{\sigma}$. Let's assume that the risky project is chosen in the case of indifference.

3. I think that the idea of this question was to get at what the risk adjusted real return in the economy, and that would be

$$R = \max[B(\sigma), \delta],$$

so $R = B(\sigma)$ for $\sigma < \hat{\sigma}$ and $R = \delta$ for $\sigma > \hat{\sigma}$.
But if we really allow a risk-free bond to be traded, then things are a bit more complicated. While it was impossible to invest in both the storage technology and the risky project at the same time, it would now be possible to invest in the riskless asset and the risky project at the same time.
Let's compute the equilibrium real interest rate under the assumption that individuals

invest in the risky project but not in storage, but may want to invest in the riskless asset. The equilibrium real interest rate must make the demand for the riskless asset equal to zero because the asset is in zero net supply.
The problem of the individual is to choose the risky investment $k_t(h)$ and investment in the riskless asset $b_t(h)$ to maximize

$$\begin{aligned} &\log(1 - k_t(h) - b_t(h)) \\ &+ \beta \left\{ \frac{1}{2} \log((A+\sigma)k_t(h) + Rb_t(h)) + \frac{1}{2} \log((A-\sigma)k_t(h) + Rb_t(h)) \right\} \end{aligned}$$

The first order conditions are

$$\begin{aligned} \frac{1}{1 - k_t(h) - b_t(h)} &= \frac{\beta}{2} \left\{ \frac{A+\sigma}{(A+\sigma)k_t(h) + Rb_t(h)} + \frac{A-\sigma}{(A-\sigma)k_t(h) + Rb_t(h)} \right\} \\ \frac{1}{1 - k_t(h) - b_t(h)} &= \frac{\beta}{2} \left\{ \frac{R}{(A+\sigma)k_t(h) + Rb_t(h)} + \frac{R}{(A-\sigma)k_t(h) + Rb_t(h)} \right\} \end{aligned}$$

We can combine them to obtain

$$\frac{A+\sigma-R}{(A+\sigma)k_t(h) + Rb_t(h)} + \frac{A-\sigma-R}{(A-\sigma)k_t(h) + Rb_t(h)} = 0$$

Now in equilibrium it must be the case that $b_t(h) = 0$, so this condition reduces to

$$\frac{A+\sigma-R}{(A+\sigma)} + \frac{A-\sigma-R}{(A-\sigma)} = 0$$

Solving this for R and expressing the result as a function of σ we get

$$R(\sigma) = \frac{(A+\sigma)(A-\sigma)}{A}.$$

Notice that $R(\sigma) < B(\sigma)$. This makes a lot of sense. If the interest rate where $B(\sigma)$, then a risk adjusted return of $B(\sigma)$ could be achieved simply by putting everything into the riskless asset. But by combining the risky project with some of the riskless asset, it would be possible to do strictly better then $B(\sigma)$. But with the interest rate $R(\sigma)$ above, it is undesirable to put anything into the riskless asset, and then the risk adjusted return is one again $B(\sigma)$. Once again this is an equilibrium only if $B(\sigma) \geq \delta$ or equivalently $\sigma \leq \hat{\sigma}$, because otherwise given the interest rate $R(\sigma)$ it would be optimal to put all savings into the storage technology (using the riskless asset would be inferior as $B(\sigma) \geq \delta$ implies $R(\sigma) < \delta$).
Now suppose that $\sigma > \hat{\sigma}$. Is it an equilibrium for all individuals to invest in storage? If so, then the interest rate must be $R = \delta$. But if the interest rate is δ, then an individual could get a risk adjusted return higher then δ by combining the riskless asset and a little bit of the risky project. So investing in storage can never be an equilibrium, and consequently an equilibrium does not exist if $\sigma > \hat{\sigma}$.
So from now on let's just consider $R(\sigma) = \max[B(\sigma), \delta]$.

4. Substituting the optimal decision between risky project and storage into the objective, the problem reduces to the maximization of

$$\log(1 - k_t(h)) + \beta \log(R(\sigma)k_t(h))$$

and the solution

$$k_t(h) = \frac{\beta}{1+\beta}$$

is independent of $R(\sigma)$.

5. If $\sigma \leq \hat{\sigma}$, then $c^o_{t+1}(h) = \tilde{A}_{t+1}(h)\frac{\beta}{1+\beta}$ and taking the average across individuals we get $\int c^o_{t+1}(h)dh = A\frac{\beta}{1+\beta}$ and as $c^y_t(h) = \frac{1}{1+\beta}$ for all individuals we get $g_t = \beta A$. On the other hand, if $\sigma > \hat{\sigma}$, then for all individuals $c^o_{t+1}(h) = \delta\frac{\beta}{1+\beta}$ and consequently $g_t = \beta\delta$. Thus as a function of σ we have

$$g(\sigma) = \begin{cases} \beta A & \text{for } \sigma \leq \hat{\sigma}, \\ \beta\delta & \text{for } \sigma > \hat{\sigma}. \end{cases}$$

6. (a) The coefficient of relative risk aversion is γ and $\frac{1}{\theta}$ is the elasticity of intertemporal substitution.

   (b) As

$$B(\sigma) = \left[\frac{1}{2}(A+\sigma)^{1-\gamma} + \frac{1}{2}(A-\sigma)^{1-\gamma}\right]^{\frac{1}{1-\gamma}}$$

   we have

$$\frac{\partial B}{\partial \sigma} = B(\sigma)^{\gamma}\frac{1}{2}[(A+\sigma)^{-\gamma} - (A-\sigma)^{-\gamma}] < 0.$$

   We would expect B to me more responsive to changes in σ if γ is large.

   (c) We would expect an increase in the coefficient of relative risk aversion to reduce the risk adjusted return, that is $\frac{\partial B}{\partial \gamma} < 0$. Recall that $\hat{\sigma}$ is implicitly defined by $B(\hat{\sigma}) = \delta$. Increasing γ reduces the left hand side, so $\hat{\sigma}$ must fall to restore equalitiy, i.e. $\frac{\partial \hat{\sigma}}{\partial \gamma} < 0$.

   (d) Now savings are chosen by maximizing

$$(1 - k_t(h))^{1-\theta} + \beta(R(\sigma)k_t(h))^{1-\theta}$$

   and the solution is

$$k_t(h) = \frac{\beta^{\frac{1}{\theta}}R(\sigma)^{\left(\frac{1}{\theta}-1\right)}}{1+\beta^{\frac{1}{\theta}}R(\sigma)^{\left(\frac{1}{\theta}-1\right)}}.$$

   Of course the saving rate is not at all sensitive to the interest rate if $\theta = 1$. For θ close to zero it is quite sensitive and it is somewhat sensitve for large θ.

(e) If $\sigma \leq \hat{\sigma}$, then $c^o_{t+1}(h) = \tilde{A}_{t+1}(h)\frac{\beta^{\frac{1}{\theta}}R(\sigma)^{\left(\frac{1}{\theta}-1\right)}}{1+\beta^{\frac{1}{\theta}}R(\sigma)^{\left(\frac{1}{\theta}-1\right)}}$ and taking the average across individuals we get $\int c^o_{t+1}(h)dh = A\frac{\beta^{\frac{1}{\theta}}R(\sigma)^{\left(\frac{1}{\theta}-1\right)}}{1+\beta^{\frac{1}{\theta}}R(\sigma)^{\left(\frac{1}{\theta}-1\right)}}$ and as $c^y_t(h) = \frac{1}{1+\beta^{\frac{1}{\theta}}R(\sigma)^{\left(\frac{1}{\theta}-1\right)}}$ for all individuals we get $g_t = A\beta^{\frac{1}{\theta}}R(\sigma)^{\left(\frac{1}{\theta}-1\right)}$. On the other hand, if $\sigma > \hat{\sigma}$, then $g_t = \beta^{\frac{1}{\theta}}\delta^{\left(\frac{1}{\theta}-1\right)}\delta$. Thus as a function of σ we have

$$g(\sigma) = \begin{cases} A\beta^{\frac{1}{\theta}}R(\sigma)^{\left(\frac{1}{\theta}-1\right)} & \text{for } \sigma \leq \hat{\sigma}, \\ (\beta\delta)^{\frac{1}{\theta}} & \text{for } \sigma > \hat{\sigma}. \end{cases}$$

The growth rate of consumption varies with σ only as long as $\sigma < \hat{\sigma}$. If $\theta < 1$, then an increase in σ will reduce $R(\sigma)$ and thus consumption growth. If $\theta > 1$, then consumption growth is increasing in σ. Of course in both cases there is still a downward jump at $\hat{\sigma}$. An increase in γ shortens the intervall $[0, \hat{\sigma}]$ in which $g(\sigma)$ varies with σ, and the counterpart of this is that within this intervall consumption growth becomes more sensitive to changes in σ

Problem 4 (Government and Growth in the Ramsey Model)
Consider a representative consumer who maximizes

$$\max \int_0^\infty e^{-\rho t}\frac{c^{1-\theta}-1}{1-\theta}dt \tag{1}$$

subject to

$$\dot{a} = (1-\tau_K)ra + (1-\tau_L)w - c + v \tag{2}$$

where c denotes consumption, a denotes assets, r is the interest rate, w is the wage rate, τ_K is the tax rate on capital income, τ_L the tax rate on labor income, and v a lump-sum per capita transfer. The government spends g per capita in order to blow up Pacific islands (i.e. g does not affect utility or production). The government budget is

$$g + v = \tau_K ra + \tau_L w$$

The market clearing for assets is

$$k = a$$

The production function is Cobb-Douglas, $y = k^\alpha$, implying $r = \alpha k^{\alpha-1}$ and $w = (1-\alpha)k^\alpha$.

1. Write down the resource constraint of the economy.

2. Write down the FOCs for maximization of the consumer, taking all fiscal policy variables as given.

3. Use a phase diagram in (k, c) to show how the paths of k and c change when the government surprises people by permanently raising the values of τ_K and g. What happens to the steady state value of k?

4. Redo part c.) for the case in which the government raises τ_L and g (without changing τ_K). What happens to the steady state value of k? Explain the differences from those of part c.)

5. Redo part c.) for the case in which the government raises τ_L and v (without changing τ_Y and g). What happens to the steady state value of k? Explain the differences from those of parts c.) and d.)

6. Finally, assume that $\tau_L = v = 0$ so that $g = \tau_K y$. Discuss the adjustment dynamics due to the following change in fiscal policy: at time $t = T_1$, the government announces that spending increases from $g = 0$ to $g = \overline{g} > 0$ until $t = T_2$.

7. Redo part f.), but now assuming the following policy change: at time $t = T_1$, the government announces that from $t = T_2 > T_1$ until $t = T_3 > T_2$ spending increases from $g = 0$ to $g = \overline{g} > 0$. What is different?

8. Discuss how your answers to all above parts would change if labor supply was endogenous.