14.451: Macroeconomic Theory I
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Handout 2: Optimization Methods in Macroeconomics

This handout summarizes the main frameworks and solution tools utilized in this course. It is designed to be a useful basic reference. For a more detailed treatment and/or applications see Professor Angeletos' notes, Professor Acemoglu's book or books on optimal control/dynamic programming, e.g. Stokey-Lucas-Prescott.

1. Continuous Time

1.1. Optimal Control: Present Value

The optimization problem is:

$$J_0 = \max_{\{x(t),y(t)\}} \int_0^T f\left(t, x(t), y(t)\right) dt$$

$$\begin{aligned} \text{s.t.} \qquad \dot{x}(t) &= g\left(t, x(t), y(t)\right) \quad \forall \;\; t \geq 0 \\ y(t) &\in \mathcal{Y}(t) \qquad\qquad\quad \forall \;\; t \geq 0 \\ x(0) &= x_0 \text{ and condition for } x(T). \end{aligned}$$

Assume the solution $y(t) \in \text{int}[\mathcal{Y}(t)] \; \forall \;\; t \geq 0$ (simple arguments can generalize this). Write the present value Hamiltonian:

$$H\left(t, x(t), y(t), \lambda(t)\right) = f\left(t, x(t), y(t)\right) + \lambda(t) g\left(t, x(t), y(t)\right) \tag{1}$$

FOCs (necessary):

$$H_y\left(t, x(t), y(t), \lambda(t)\right) = 0 : \quad f_y\left(t, x(t), y(t)\right) + \lambda(t) g_y\left(t, x(t), y(t)\right) = 0 \tag{2}$$

$$H_x\left(t, x(t), y(t), \lambda(t)\right) = -\dot{\lambda}(t) : \quad f_x\left(t, x(t), y(t)\right) + \lambda(t) g_x\left(t, x(t), y(t)\right) = -\dot{\lambda}(t) \tag{3}$$

$$H_\lambda\left(t, x(t), y(t), \lambda(t)\right) = \dot{x}(t) : \quad \dot{x}(t) = g\left(t, x(t), y(t)\right) \tag{4}$$

$$x(0) = x_0 \text{ and (necessary) transversality conditions.} \tag{5}$$

For finite horizon $T < \infty$:
Different conditions for $x(T)$ imply different transversality conditions:

Type of problem	Boundary condition on problem	Transversality condition for solution
Vertical terminal line	$x(T)$ is free	$\lambda(T) = 0$
Truncated vertical terminal line	$x(T) \geq x_T$	$\lambda(T) \geq 0, x(T) \geq x_T$ with $\left(x(T) - x_T\right) \lambda(T) = 0$
Horizontal terminal line	$x(T) = x_T$, T is free	$H\left(T, x(T), y(T), \lambda(T)\right) = 0$
Fixed terminal point	$x(T) = x_T$, T given	$x(T) = x_T$

For infinite horizon $T = \infty$:
There is no general transversality condition. Examples of strong transversality conditions inspired by finite horizon conditions:

$$\begin{aligned} \lim_{t \to \infty} \lambda(t) &= 0. \\ \lim_{t \to \infty} \lambda(t) &\geq 0, \lim_{t \to \infty} x(t) \geq x_\infty \text{ with } \lim_{t \to \infty} \left(x(t) - x_\infty\right) \lambda(t) = 0 \end{aligned}$$

Weaker transversality condition that holds for some problems where the objective function is normalized around the steady state:

$$\lim_{t \to \infty} H\left(t, x(t), y(t), \lambda(t)\right) = 0$$

Mangasarian conditions (sufficient): Necessary conditions satisfied plus:

$$H\left(t, x(t), y(t), \lambda(t)\right) \text{ jointly concave in } \left(x(t), y(t)\right) \text{ for the } \lambda(t) \text{ from FOCs}$$

$$\lim_{t\to\infty} \lambda(t)\left(x(t) - \tilde{x}(t)\right) \leq 0 \;\; \forall\; \tilde{x}(t) \text{ implied by admissible control paths } y(t) \in \mathcal{Y}(t)$$

Then FOCs and TVC are necessary and sufficient.

1.2. Optimal Control: Current Value

The optimization problem is:

$$J_0 = \max_{\{x(t), y(t)\}} \int_0^\infty \exp\left(-\rho t\right) f\left(x(t), y(t)\right) dt \quad \text{with } \rho > 0$$

$$\begin{array}{lrcl} \text{s.t.} & \dot{x}(t) & = & g\left(x(t), y(t)\right) \quad \forall \;\; t \geq 0 \\ & y(t) & \in & \mathcal{Y}(t) \qquad\qquad \forall \;\; t \geq 0 \\ & x(0) & = & x_0 \text{ and } \lim_{t\to\infty} x(t) \geq \bar{x}. \end{array}$$

Write the current value Hamiltonian:

$$\hat{H}\left(x(t), y(t), \mu(t)\right) = f\left(x(t), y(t)\right) + \mu(t) g\left(x(t), y(t)\right) \tag{6}$$

FOCs (necessary):

$$\hat{H}_y\left(x(t), y(t), \mu(t)\right) = 0 : \quad f_y\left(x(t), y(t)\right) + \mu(t) g_y\left(x(t), y(t)\right) = 0 \tag{7}$$

$$\hat{H}_x\left(x(t), y(t), \mu(t)\right) = -\dot{\mu}(t) + \rho\mu(t) : \quad f_x\left(x(t), y(t)\right) + \mu(t) g_x\left(x(t), y(t)\right) = -\dot{\mu}(t) + \rho\mu(t) \tag{8}$$

$$\hat{H}_\mu\left(x(t), y(t), \mu(t)\right) = \dot{x}(t) : \quad \dot{x}(t) = g\left(x(t), y(t)\right) \tag{9}$$

$$x(0) = x_0 \tag{10}$$

$$\text{Check } y(t) \in \mathcal{Y}(t). \tag{11}$$

Transversality condition (necessary):

$$\lim_{t\to\infty} \left[\exp\left(-\rho t\right) x(t) \mu(t)\right] = 0 \tag{12}$$

for a range of $x(t)$ such that f, g are weakly monotone.
Mangasarian conditions (sufficient): Necessary conditions satisfied plus:

$$\hat{H}\left(x(t), y(t), \mu(t)\right) \text{ jointly concave in } \left(x(t), y(t)\right) \text{ for the } \mu(t) \text{ from FOCs}$$

Then FOCs and TVC are necessary and sufficient.

2. Discrete Time

The optimization problem is:

$$U_0 = \max_{\{c_t, l_t, k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t U(c_t, 1 - l_t)$$

$$\begin{aligned} \text{s.t.} \qquad c_t + k_{t+1} &\leq F(k_t, l_t) + (1-\delta)\, k_t \quad \forall\ t \geq 0 \\ c_t, l_t, 1-l_t, k_{t+1} &\geq 0 \qquad\qquad\qquad\qquad\ \ \forall\ t \geq 0 \\ k_0 &\geq 0 \text{ given.} \end{aligned}$$

2.1. Optimal Control

Rewrite the problem as a standard optimization problem with discounted Lagrangian multipliers:

$$\max_{\{c_t, l_t, k_{t+1}\}_{t=0}^{T}} \mathcal{L} = \sum_{t=0}^{T} \beta^t U(c_t, 1 - l_t) + \sum_{t=0}^{T} \beta^t \lambda_t \left[(1-\delta)\, k_t + F(k_t, l_t) - k_{t+1} - c_t\right]$$

FOCs (rewrite $z_t = 1 - l_t$):

$$c_t: \ \beta^t U_c(c_t, z_t) - \beta^t \lambda_t = 0 \Leftrightarrow U_c(c_t, z_t) = \lambda_t \tag{13}$$

$$l_t: \ -\beta^t U_z(c_t, 1-l_t) + \beta^t \lambda_t F_l(k_t, l_t) = 0 \Leftrightarrow U_z(c_t, 1-l_t) = \lambda_t F_l(k_t, l_t) \tag{14}$$

$$k_{t+1}: \ -\beta^t \lambda_t + \beta^{t+1} \lambda_{t+1} \left[1 - \delta + F_k(k_{t+1}, l_{t+1})\right] = 0 \Leftrightarrow \lambda_t = \beta \lambda_{t+1} \left[1 - \delta + F_k(k_{t+1}, l_{t+1})\right] \tag{15}$$

EC:

$$k_t: \ \frac{\partial U_0}{\partial k_0} = \frac{\partial \mathcal{L}}{\partial k_0} = \lambda_0 = U_c(c_0, z_0) \tag{16}$$

FOCs and TVC are necessary. They are also sufficient if the problem is convex.
From this we derive the following.
Intratemporal: Consumption versus Labor. Equation (14) divided by equation (13):

$$\frac{U_z(c_t, 1-l_t)}{U_c(c_t, 1-l_t)} = F_l(k_t, l_t) \tag{17}$$

Intertemporal: Consumption Euler. Substitute equation (13) into equation (15) and rearrange:

$$\frac{U_c(c_t, 1-l_t)}{U_c(c_{t+1}, 1-l_{t+1})} = \beta \left[1 - \delta + F_k(k_{t+1}, l_{t+1})\right] \tag{18}$$

Transversality Condition: For finite horizon $T < \infty$:

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial k_{T+1}} &\leq 0, k_{T+1} \geq 0 \ \text{ with complementary slackness} \\ &\Leftrightarrow -\beta^T \lambda_T \leq 0, k_{T+1} \geq 0 \ \text{ with complementary slackness} \end{aligned}$$

$$\Leftrightarrow \beta^T U_c(c_T, 1-l_T) \geq 0, k_{T+1} \geq 0 \ \text{ with } \beta^T U_c(c_T, 1-l_T) k_{T+1} = 0 \tag{19}$$

For infinite horizon $T = \infty$:

$$\lim_{t \to \infty} \beta^t U_c(c_t, 1-l_t) k_{t+1} = 0 \tag{20}$$

2.2. Dynamic Programming

Rewrite the problem using a Bellman equation formulation, where a prime $'$ on a variable indicates the value of the variable in the next period:

$$V(k) = \max_{c,l,k} \{U(c, 1-l) + \beta V(k')\}$$

$$\begin{array}{lrcl} \text{s.t.} & c + k' & \leq & F(k,l) + (1-\delta)\,k \\ & k' & \in & [0, F(k,l) + (1-\delta)\,k],\ c \in [0, F(k,l) + (1-\delta)\,k],\ l \in [0,1] \\ & k_0 & \geq & 0 \text{ given.} \end{array}$$

Assume that $V(k)$ is continuous, differentiable and strictly concave. We can write the policy functions as $c_t = c(k_t)$, $l_t = l(k_t)$, $k_{t+1} = G(k_t)$. The Euler and transversality conditions are necessary and sufficient for a maximum.

$$\max_{c,l,k'} \mathcal{L} = U(c, 1-l) + \beta V(k') + \lambda\,[(1-\delta)k + F(k,l) - k' - c]$$

FOCs:

$$c:\ U_c(c,z) - \lambda = 0 \Leftrightarrow U_c(c,z) = \lambda \tag{21}$$

$$l:\ -U_z(c,z) + \lambda F_l(k,l) = 0 \Leftrightarrow U_z(c,z) = \lambda F_l(k,l) \tag{22}$$

$$k':\ \beta V_k(k') - \lambda = 0 \Leftrightarrow \beta V_k(k') = \lambda \tag{23}$$

EC:

$$k:\ V_k(k) = \frac{\partial \mathcal{L}}{\partial k} = \lambda\,[1 - \delta + F_k(k,l)] \tag{24}$$

From this we derive the following.
Intratemporal: Consumption versus Labor. Equation (22) divided by equation (21):

$$\frac{U_z(c_t, 1-l_t)}{U_c(c_t, 1-l_t)} = F_l(k_t, l_t) \tag{25}$$

Intertemporal: Consumption Euler. Equation (24) implies:

$$V_k(k') = \lambda'\,[1 - \delta + F_k(k', l')] \tag{26}$$

Divide equation (23) by equation (26), then rearrange:

$$\frac{\lambda}{\lambda'} = \beta\,[1 - \delta + F_k(k', l')] \tag{27}$$

Substitute equation (21) into equation (27):

$$\frac{U_c(c_t, 1-l_t)}{U_c(c_{t+1}, 1-l_{t+1})} = \beta\,[1 - \delta + F_k(k_{t+1}, l_{t+1})] \tag{28}$$

Intertemporal: Capital Euler. Equation (24) divided by equation (23):

$$\frac{V_k(k_t)}{V_k(k_{t+1})} = \beta\,[1 - \delta + F_k(k_t, l_t)] \tag{29}$$

Transversality Condition:

$$\lim_{t\to\infty} \beta^t U_c(c_t, 1-l_t) k_{t+1} = 0 \tag{30}$$